

Differential modules on p -adic polyannuli

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Abstract

We consider variational properties of some numerical invariants, measuring convergence of local horizontal sections, associated to differential modules on polyannuli over a nonarchimedean field of characteristic zero. This extends prior work in the one-dimensional case of Christol, Dwork, Robba, Young, et al. Our results do not require positive residue characteristic; thus besides their relevance to the study of Swan conductors for isocrystals, they are germane to the formal classification of flat meromorphic connections on complex manifolds.

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Introduction

Differential equations involving p -adic analytic functions have a nasty habit of failing to admit global solutions even in the absence of singularities; for instance, the exponential series fails to be entire. To measure this, Dwork and his collaborators introduced the notion of the *generic radius of convergence* of a p -adic differential module over a one-dimensional space (for simplicity, we restrict attention here to discs and annuli). The modern definition of this concept was given and studied in depth by Christol and Dwork [3]. A further refinement, the collection of *subsidiary generic radii of convergence*, was introduced (under different terminology) by Young [20].

Given a differential module over a p -adic disc or annulus of the form $\alpha \leq |t| \leq \beta$, one obtains a generic radius of convergence and some subsidiary radii for each radius $\rho \in [\alpha, \beta]$, and one would like to be able to say something about how these quantities vary with ρ . (In fact, one also obtains these data for each point of the Berkovich1 analytic space; this is the point of view adopted in ongoing work of Baldassarri and di Vizio, starting with [1].) By pulling together techniques from the literature and adding one or two new ideas, one can make fairly definitive statements about the nature of this variation; this was done by the first author in a course given in fall 2007, whose compiled notes constitute the volume [11].

The course [11] was deliberately restricted to the study of p -adic *ordinary* differential equations. One could view the extension of the variational results to higher-dimensional spaces as an implied exercise in [11]. This paper constitutes a partial solution of this implied exercise, in which we obtain variational properties for differential modules over certain higher-dimensional p -adic analytic spaces. The spaces we consider are what one might call *generalized polyannuli*: such a space is an analytic subspace of an affine space in some variables t_1, \dots, t_n , defined by the restriction $(|t_1|, \dots, |t_n|) \in S$ for some set S such that $\log S$ is convex. (In order for this to actually define an analytic space, one must impose some

polyhedrality conditions on $\log S$. For an example of what happens when such conditions are missing, see the treatment of “fake annuli” in [9].)

The strategy we adopt is to proceed in three stages. We start with some formalism for differential modules over differential fields (corresponding to zero-dimensional spaces), in somewhat greater generality than in [11]. We then make a series of calculations on a one-dimensional annulus over a nonarchimedean field which itself carries one or more commuting derivations. We consider modules equipped with commuting actions of both the base derivations and the derivation in the geometric direction, and obtain results in the spirit of those in [11]. We finally extend these results to higher-dimensional spaces (which may still carry derivations on the base field) by using some careful analysis of convex functions on polyhedral subsets of \mathbb{R}^n .

The original intended application of these results is to the study of differential Swan conductors for isocrystals, as introduced by the first author in [8]. (The extra work in Section 1 is needed to obtain a common generalization of the hypotheses in [11] and [8].) The deployment of these results in the study of differential Swan conductors takes place in [12], following up on earlier investigations by Matsuda [15]. Since our results do not require positive residual characteristic, they are also relevant to formal classification of flat meromorphic connections on complex manifolds, as in the work of Sabbah [18] for complex analytic surfaces. For instance, the first author [13] recently used the results of this paper to resolve the main conjecture of [18].

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1 Differential modules over a field

In this section, we assemble a slightly more comprehensive collection of definitions and basic results concerning differential modules over a field than was given in [7, §1]. This is done in order to state results applicable in the context of [8].

1.1 Setup

Convention 1.1.1. Let $f^* : R_1 \rightarrow R_2$ be a homomorphism of rings. For an R_1 -module M_1 , we write f^*M_1 to denote the extension of scalars $M_1 \otimes_{R_1, f^*} R_2$. For an R_2 -module M_2 we write f_*M_2 to mean M_2 viewed as an R_1 -module via f^* (i.e., the restriction of scalars).

Convention 1.1.2. For any nonarchimedean field K of characteristic zero, denote its ring of integers and residue field by \mathfrak{o}_K and k , respectively. We reserve the letter p for the residual characteristic of K . If $p > 0$, we normalize the norm $|\cdot|$ on K so that $|p| = 1/p$. For an element $a \in \mathfrak{o}_K$, we denote its reduction in k by \bar{a} . In case K is discretely valued, let π_K denote a uniformizer of \mathfrak{o}_K .

Definition 1.1.3. A finite extension L of a complete nonarchimedean field K is *unramified* if L and K have the same value group, and the residue field extension is separable of degree $[L : K]$. It is *tamely ramified* if the index $e_{L/K}$ of the value group of K in that of L is not divisible by p , and the residue field extension is separable of degree $[L : K]/e_{L/K}$; we call $e_{L/K}$ the *ramification degree*. If $p = 0$, then any finite extension of K is tamely ramified, by a theorem of Ostrowski (see [17, Chapter 6]). For L the completion of an infinite algebraic extension of K , we say that L is unramified or tamely ramified if the same is true of each finite subextension of L over K ; we define the ramification degree to be the supremum of the ramification degrees of the finite subextensions.

Convention 1.1.4. Let J be a finite index set. We will write e_J for a tuple $(e_j)_{j \in J}$. For another tuple u_J , write $u_J^{e_J} = \prod_{j \in J} u_j^{e_j}$. We also use $\sum_{e_J=0}^n$ to mean the sum over $e_j \in \{0, 1, \dots, n\}$ for each $j \in J$; for notational simplicity, we may suppress the range of the summation when it is clear. Write $|e_J| = \sum_{j \in J} |e_j|$ and $(e_J)!$ for $\prod_{j \in J} (e_j)!$.

Convention 1.1.5. For a matrix $A = (A_{ij})$ with coefficients in a nonarchimedean ring, we use $|A|$ to denote the supremum norm over entries.

Hypothesis 1.1.6. For the rest of this subsection, we assume that K is a complete nonarchimedean field.

Notation 1.1.7. Let $I \subset [0, +\infty)$ be an interval. Let $A_K^1(I)$ denote the annulus with radii in I . (We do not impose any rationality condition on the endpoints of I , so this space should be viewed as an analytic space in the sense of Berkovich [2].) If I is written explicitly in terms of its endpoints (e.g., $[\alpha, \beta]$), we suppress the parentheses around I (e.g., $A_K^1[\alpha, \beta]$). For $0 \leq \alpha \leq \beta < \infty$, let $K\langle \alpha/t, t/\beta \rangle$ denote the ring of analytic functions on $A_K^1[\alpha, \beta]$. (If $\alpha = 0$, we write $K\langle t/\beta \rangle$ instead.)

Definition 1.1.8. We have the ring of *series with bounded coefficients*

$$K[[t/\beta]]_0 = \left\{ \sum_{i=0}^{\infty} a_i t^i \in K[[t]] : \sup_i \{|a_i| \beta^i\} < \infty \right\};$$

these are the power series which converge and take bounded values on the open disc $|t| < \beta$. Note that for any $\delta \in (0, \beta)$,

$$K\langle t/\beta \rangle \subset K[[t/\beta]]_0 \subset K\langle t/\delta \rangle.$$

In particular, when $\beta = 1$, we have

$$K[[t]]_0 = \mathfrak{o}_K[[t]] \otimes_{\mathfrak{o}_K} K.$$

An analogue of this construction for an annulus is

$$K\langle \alpha/t, t/\beta \rangle_0 = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i : a_i \in K, \lim_{i \rightarrow -\infty} |a_i| \alpha^i = 0, \sup_i \{|a_i| \beta^i\} < \infty \right\};$$

these are the Laurent series which converge and take bounded values on the half-open annulus $\alpha \leq |t| < \beta$. For any $\delta \in [\alpha, \beta)$, this ring satisfies

$$K\langle \alpha/t, t/\beta \rangle \subset K\langle \alpha/t, t/\beta \rangle_0 \subset K\langle \alpha/t, t/\delta \rangle.$$

Definition 1.1.9. Define the ring

$$K\{\{t/\beta\}\} = \bigcap_{\delta \in (0, \beta)} K\langle t/\delta \rangle = \left\{ \sum_{i=0}^{\infty} a_i t^i : a_i \in K, \lim_{i \rightarrow \infty} |a_i| \rho^i = 0 \quad (\rho \in (0, \beta)) \right\};$$

these are the power series convergent on the open disc $|t| < \beta$, with no boundedness restriction. In particular, for any $\delta \in (0, \beta)$,

$$K\llbracket t/\beta \rrbracket_0 \subset K\{\{t/\beta\}\} \subset K\langle t/\delta \rangle.$$

An analogue of the previous construction for an annulus is

$$K\{\{\alpha/t, t/\beta\}\} = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i : a_i \in K, \lim_{i \rightarrow \pm \infty} |a_i| \eta^i = 0 \quad (\eta \in (\alpha, \beta)) \right\};$$

these are the Laurent series convergent on the open annulus $\alpha < |t| < \beta$.

Definition 1.1.10. Put $I = \{1, \dots, n\}$. For $(\eta_i)_{i \in I} \in (0, +\infty)^n$, the η_I -Gauss norm on $K[t_I]$ is the norm $|\cdot|_{\eta_I}$ given by

$$\left| \sum_{e_I} a_{e_I} t_I^{e_I} \right|_{\eta_I} = \max \{ |a_{e_I}| \cdot \eta_I^{e_I} \};$$

this norm extends uniquely to $K(t_I)$.

For $\eta \in [\alpha, \beta]$ and $\eta \neq 0$, let $x = \sum_{i=-\infty}^{\infty} a_i t^i$ be an element of $K\langle \alpha/t, t/\beta \rangle$, $K\langle \alpha/t, t/\beta \rangle_0$, or (if $\eta \neq \alpha, \beta$) $K\{\{\alpha/t, t/\beta\}\}$. We define the η -Gauss norm of x to be

$$|x|_{\eta} = \sup \{ |a_i| \cdot \eta^i \}.$$

Convention 1.1.11. By a G -map, we will mean a morphism of affinoid or Stein (K -)analytic spaces with G -topology, which need not respect the K -space structure. This amounts to a homomorphism between the corresponding rings of global sections, which need not be K -linear. For example, the homomorphism f_{gen}^* defined in Lemma 1.2.12 below gives rise to a G -map $f_{\text{gen}} : A_K^1[0, R_{\partial}(K)) \rightarrow \text{Max}(K)$.

Convention 1.1.12. Throughout this paper, all derivations on topological modules will be assumed to be continuous; moreover, any derivation considered on a ring equipped with a nonarchimedean norm will be assumed to be bounded (i.e., to have bounded operator norm). All connections considered will be assumed to be integrable. We may suppress the base ring from a module of continuous differentials when it is unambiguous.

1.2 Differential fields and differential modules

Definition 1.2.1. Let K be a differential ring of order 1, i.e., a ring equipped with a derivation ∂ . Let $K\{T\}$ denote the (noncommutative) ring of twisted polynomials over K [16]; its elements are finite formal sums $\sum_{i \geq 0} a_i T^i$ with $a_i \in K$, multiplied according to the rule $Ta = aT + \partial(a)$ for $a \in K$.

Definition 1.2.2. A ∂ -differential module over K is a finite projective K -module V equipped with an action of ∂ (subject to the Leibniz rule); any ∂ -differential module over K inherits a left action of $K\{T\}$ where T acts via ∂ . The module dual $V^\vee = \text{Hom}_K(V, K)$ of V may be viewed as a ∂ -differential module by setting $(\partial f)(\mathbf{v}) = \partial(f(\mathbf{v})) - f(\partial(\mathbf{v}))$. We say V is *free* if V as a module is free over K . We say V is *trivial* if it is free and there exists a K -basis $\mathbf{v}_1, \dots, \mathbf{v}_d \in V$ such that $\partial(\mathbf{v}_i) = 0$ for $i = 1, \dots, d$.

For V a differential module over K , we say $\mathbf{v} \in V$ is a *cyclic vector* if $\mathbf{v}, \partial\mathbf{v}, \dots, \partial^{\text{rank}(V)-1}\mathbf{v}$ form a basis of V . A cyclic vector defines an isomorphism $V \simeq K\{T\}/K\{T\}P$ of differential modules for some twisted polynomial $P \in K\{T\}$, where the ∂ -action on $K\{T\}/K\{T\}P$ is the left multiplication by T .

Definition 1.2.3. For a differential module V over K , define

$$H_\partial^0(V) = \text{Ker } \partial, \quad H_\partial^1(V) = \text{Coker } \partial = V/\partial(V).$$

The latter computes Yoneda extensions; see, e.g., [11, Lemma 5.3.3].

Lemma 1.2.4. *If K is a field of characteristic zero, every differential module over K contains a cyclic vector.*

Proof. See, e.g., [4, Theorem III.4.2] or [11, Theorem 5.4.2]. □

Hypothesis 1.2.5. For the rest of this subsection, we assume that K is a field of characteristic zero complete for a nonarchimedean norm $|\cdot|$ and equipped with a derivation ∂ with operator norm $|\partial|_K < \infty$, and that V is a nonzero ∂ -differential module over K .

Definition 1.2.6. The *spectral norm of ∂ on V* is defined to be

$$|\partial|_{\text{sp}, V} = \lim_{n \rightarrow \infty} |\partial^n|_V^{1/n}$$

for any fixed K -compatible norm $|\cdot|_V$ on V . Any two such norms on V are equivalent [19, Proposition 4.13], so the spectral norm does not depend on the choice [11, Proposition 6.1.5]. One can show that $|\partial|_{\text{sp}, V} \geq |\partial|_{\text{sp}, K}$ [11, Lemma 6.2.4].

Explicitly, if one chooses a basis of V and lets D_n denote the matrix via which ∂^n acts on this basis, then

$$|\partial|_{\text{sp}, V} = \max\{|\partial|_{\text{sp}, K}, \lim_{n \rightarrow \infty} |D_n|^{1/n}\}.$$

Remark 1.2.7. If $K \rightarrow K'$ is an isometric embedding of complete nonarchimedean differential fields, then for a differential module V over K , $V' = V \otimes_K K'$ is a differential module over K' , and $|\partial|_{\text{sp}, V'} = \max\{|\partial|_{\text{sp}, K'}, |\partial|_{\text{sp}, V}\}$.

Definition 1.2.8. Let p denote the residual characteristic of K ; we conventionally write

$$\omega = \begin{cases} 1 & p = 0 \\ p^{-1/(p-1)} & p > 0 \end{cases}.$$

Define the *generic ∂ -radius of convergence* (or for short, the *generic ∂ -radius*) of V to be

$$R_\partial(V) = \omega |\partial|_{\text{sp}, V}^{-1};$$

note that $R_\partial(V) > 0$. We will see later (Proposition 1.2.14) that this indeed computes the radius of convergence of Taylor series on a “generic disc”. In some situations, it is more natural to consider the *intrinsic generic ∂ -radius of convergence*, or for short the *intrinsic ∂ -radius*, defined as

$$IR_\partial(V) = \frac{|\partial|_{\text{sp}, K}}{|\partial|_{\text{sp}, V}};$$

note that this is a number in $(0, 1]$.

Let V_1, \dots, V_d be the Jordan-Hölder constituents of V . We define the (*extrinsic*) *subsidiary generic ∂ -radii of convergence*, or for short the *subsidiary ∂ -radii*, to be the multiset $\mathfrak{R}_\partial(V)$ consisting of $R_\partial(V_i)$ with multiplicity $\dim V_i$ for $i = 1, \dots, d$. Let $R_\partial(V; 1) \leq \dots \leq R_\partial(V; \dim V)$ denote the elements in $\mathfrak{R}_\partial(V)$ in increasing order. We similarly define *intrinsic subsidiary (generic) ∂ -radii of convergence* $\mathfrak{IR}_\partial(V)$, or for short *intrinsic subsidiary ∂ -radii*, by aggregating the intrinsic ∂ -radii of V_i for $i = 1, \dots, d$. Let $IR_\partial(V; 1) \leq \dots \leq IR_\partial(V; \dim V)$ denote the elements in $\mathfrak{IR}_\partial(V)$ in increasing order.

We say that V has *pure ∂ -radii* if $\mathfrak{R}(V)$ consists of d copies of $R_\partial(V)$.

Lemma 1.2.9. Let V, V_1, V_2 be nonzero ∂ -differential modules over K .

(a) For $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ exact,

$$R_\partial(V) = \min \{R_\partial(V_1), R_\partial(V_2)\}; \quad IR_\partial(V) = \min \{IR_\partial(V_1), IR_\partial(V_2)\}.$$

More precisely,

$$\mathfrak{R}_\partial(V) = \mathfrak{R}_\partial(V_1) \cup \mathfrak{R}_\partial(V_2); \quad \mathfrak{IR}_\partial(V) = \mathfrak{IR}_\partial(V_1) \cup \mathfrak{IR}_\partial(V_2).$$

(b) We have

$$\begin{aligned} R_\partial(V^\vee) &= R_\partial(V); & IR_\partial(V^\vee) &= IR_\partial(V); \\ \mathfrak{R}_\partial(V^\vee) &= \mathfrak{R}_\partial(V); & \mathfrak{IR}_\partial(V^\vee) &= \mathfrak{IR}_\partial(V); \end{aligned}$$

(c) We have

$$R_\partial(V_1 \otimes V_2) \geq \min \{R_\partial(V_1), R_\partial(V_2)\}; \quad IR_\partial(V_1 \otimes V_2) \geq \min \{IR_\partial(V_1), IR_\partial(V_2)\},$$

with equality when $R_\partial(V_1) \neq R_\partial(V_2)$, or equivalently, when $IR_\partial(V_1) \neq IR_\partial(V_2)$.

(d) If V_1 and V_2 are irreducible and $IR_\partial(V_1) \neq IR_\partial(V_2)$, then $\mathfrak{IR}_\partial(V_1 \otimes V_2)$ is just $\dim V_1 \cdot \dim V_2$ copies of $\min \{IR_\partial(V_1), IR_\partial(V_2)\}$

Proof. As in [11, Lemma 6.2.8] and [11, Corollary 6.2.9]. \square

Definition 1.2.10. Let R be a complete K -algebra. For $\mathbf{v} \in V$ and $x \in R$, define the ∂ -Taylor series to be

$$\mathbb{T}(\mathbf{v}; \partial, x) = \sum_{n=0}^{\infty} \frac{\partial^n(\mathbf{v})}{n!} x^n \in V \widehat{\otimes}_K R$$

in case this series converges.

Remark 1.2.11. If $V = K$, the ∂ -Taylor series gives a ring homomorphism $K \rightarrow R$ if it converges. For general V , the ∂ -Taylor series gives a homomorphism of modules $V \rightarrow V \widehat{\otimes}_K R$ via the aforementioned ring homomorphism, if it converges.

Lemma 1.2.12. *The Taylor series $x \mapsto \mathbb{T}(x; \partial, T)$ gives a continuous homomorphism $f_{\text{gen}}^* : K \rightarrow K[[T/R_{\partial}(K)]]_0$, which induces a G -map $f_{\text{gen}} : A_K^1[0, R_{\partial}(K)) \rightarrow \text{Max}(K)$. Moreover, for $\eta \in [0, R_{\partial}(K)]$, f_{gen}^* is isometric for the η -Gauss norm on the target.*

Proof. It is straightforward to check that f_{gen}^* is bounded for the η -Gauss norm for any $\eta \in [0, R_{\partial}(K))$; that is, there exists $c > 0$ such that for all $x \in K$, $|f_{\text{gen}}^*(x)|_{\eta} \leq c|x|$. For any positive integer n , we can plug x^n into the previous inequality to deduce $|f_{\text{gen}}^*(x)|_{\eta} \leq c^{1/n}|x|$. Consequently, $|f_{\text{gen}}^*(x)|_{\eta} \leq |x|$ for any $\eta \in [0, R_{\partial}(K))$, and by continuity also for $\eta = R_{\partial}(K)$. \square

Corollary 1.2.13. *For each positive integer n , we have $|\partial^n/n!|_K \leq R_{\partial}(K)^{-n} = \omega^{-n}|\partial|_{\text{sp}, K}^n$. In particular (by taking $n = 1$), $|\partial|_{\text{sp}, K} \geq \omega|\partial|_K$.*

We have the following geometric interpretation of generic radii. This is slightly different from, but essentially equivalent to, the treatments in [8, Section 2.2] and [11, Section 9.7].

Proposition 1.2.14. *With notation as in Lemma 1.2.12, the pullback f_{gen}^*V becomes a ∂_T -differential module over $A_K^1[0, R_{\partial}(K))$, where $\partial_T = \frac{d}{dT}$. Then for any $r \in (0, R_{\partial}(K))$, $R_{\partial}(V) \geq r$ if and only if f_{gen}^*V restricts to a trivial ∂_T -differential module over $A_K^1[0, r)$.*

Proof. Since f_{gen}^* is an isometry and $|\partial_T|_{K[[T/R_{\partial}(K)]]_0} = R_{\partial}(K)^{-1}$, we have $R_{\partial}(V) = R_{\partial_T}(f_{\text{gen}}^*V \otimes \text{Frac } K[[T/R_{\partial}(K)]]_0)$. It then suffices to check that $R_{\partial_T}(f_{\text{gen}}^*V) \geq r$ if and only if f_{gen}^*V restricts to a trivial ∂_T -differential module over $A_K^1[0, r)$; this is the content of Dwork's transfer theorem [11, Theorem 9.6.1]. \square

1.3 Newton polygons

In this subsection, we summarize some results in [11, Chapter 5 and 6] and [8, Section 1]. Throughout this subsection, let K be a complete nonarchimedean differential field of characteristic zero.

Definition 1.3.1. For $P(T) = \sum_i a_i T^i \in K\{T\}$ a nonzero twisted polynomial, define the *Newton polygon* of P as the lower convex hull of the set $\{(-i, -\log |a_i|)\} \subset \mathbb{R}^2$. This Newton polygon obeys the usual additivity rules only for slopes less than $-\log |\partial|_K$.

Proposition 1.3.2 (Christol-Dwork). *Suppose $V \simeq K\{T\}/K\{T\}P$, and let s be the lesser of $-\log |\partial|_K$ and the least slope of P . Then*

$$\max\{|\partial|_K, |\partial|_{\text{sp}, V}\} = e^{-s}.$$

Proof. See [3, Théorème 1.5] or [11, Theorem 6.5.3]. □

Proposition 1.3.3 (Robba). *Any monic twisted polynomial $P \in K\{T\}$ admits a unique factorization*

$$P = P_+ P_n \cdots P_1$$

such that for some $s_1 < \cdots < s_n < -\log |\partial|_K$, each P_i is monic with all slopes equal to s_i , and P_+ is monic with all slopes at least $-\log |\partial|_K$.

Proof. See [8, Proposition 1.1.10] or [10, Corollary 3.2.4]. □

Proposition 1.3.4. *Suppose that $\omega \cdot |\partial|_K^{-1} = r_0$. Then there is a unique decomposition*

$$V = V_+ \oplus \bigoplus_{r < r_0} V_r$$

of differential modules, such that V_r has pure ∂ -radii r , and the subsidiary radii of V_+ are all at least r_0 .

Proof. Apply Lemma 1.2.4 to write $V \simeq K\{T\}/K\{T\}P$ for P a twisted polynomial. Then the statement may be deduced from Proposition 1.3.3, applied first to P in $K\{T\}$ and then to P in the opposite ring. For more details, one may consult [11, Theorem 6.6.1]. □

Remark 1.3.5. If $V \simeq K\{T\}/K\{T\}P$ for P a twisted polynomial, then Propositions 1.3.2 and 1.3.3 imply that the multiplicity of any $s < -\log |\partial|_K$ as a slope of the Newton polygon of P coincides with the multiplicity of ωe^s in $\mathcal{R}_\partial(V)$.

1.4 Moving along Frobenius

As discovered originally by Christol-Dwork [3], and amplified by the first author [11], in the situation of Definition 1.4.1, one can overcome the limitation on subsidiary radii imposed by Proposition 1.3.2 by using the pushforward along the Frobenius. In this subsection, we imitate the techniques in [11, Chapter 10] and obtain Theorems 1.4.19 and 1.4.21 as analogues of [11, Theorems 10.5.1 and 10.6.2].

Definition 1.4.1. Let K be a complete nonarchimedean differential field of characteristic zero and residual characteristic p . The derivation ∂ on K is of *rational type* if there exists $u \in K$ such that the following conditions hold. (If these hold, we call u a *rational parameter* for ∂ .)

- (a) We have $\partial(u) = 1$ and $|\partial|_K = |u|^{-1}$.

(b) For each positive integer n , $|\partial^n/n!|_K \leq |\partial|_K^n$.

It is equivalent to formulate (b) as follows.

(b') We have $|\partial|_{\text{sp},K} \leq \omega|\partial|_K$.

(It is clear that (b) implies (b'); the reverse implication holds by Corollary 1.2.13.) For $p > 0$, in the presence of (a), yet another equivalent formulation of (b) is as follows.

(b'') For each polynomial $P \in \mathbb{Q}_p[T]$ such that $P(\mathbb{Z}_p) \subseteq \mathbb{Z}_p$, $|P(u\partial)|_K \leq 1$.

This relies on the fact that the \mathbb{Z}_p -module of such P is freely generated by the binomial polynomials

$$\binom{T}{n} = \frac{T(T-1)\cdots(T-n+1)}{n!} \quad (n = 0, 1, \dots).$$

Remark 1.4.2. Note that in Definition 1.4.1, the inequality in (b') is forced to be an equality by Corollary 1.2.13, while the inequality in (b) is forced to be an equality if (a) holds because then $(\partial^n/n!)(u^n) = 1$. In particular, for any nonzero ∂ -differential module V , $IR_\partial(V) = |u| \cdot R_\partial(V)$. Similarly, if (a) holds and $p > 0$, then the inequality in (b'') becomes an equality whenever $P(\mathbb{Z}_p) \not\subseteq p\mathbb{Z}_p$.

Remark 1.4.3. If u' is a second rational parameter for ∂ , then $u - u' \in \ker(\partial)$ and $|u - u'| \leq |u|$. The converse is also true; that is, if u is a rational parameter, $u - u' \in \ker(\partial)$, and $|u - u'| \leq |u|$, then u' is also a rational parameter. The only nonobvious part of this statement is the fact that these two conditions imply $|u'| = |u|$. It is clear that $|u'| \leq |u|$; on the other hand, since $\partial(u') = 1$, $1 \leq |\partial|_K |u'| = |u'|/|u|$, so $|u'| \geq |u|$.

Remark 1.4.4. The simplest case of Definition 1.4.1 is the derivation d/dt on the completion of the rational function field $\mathbb{Q}_p(t)$ for any Gauss norm if $p > 0$, or on the ring of Laurent series $\mathbb{C}((t))$ if $p = 0$. For more cases, see Situation 1.5.8 and the following remarks.

Lemma 1.4.5. *Let L/K be a complete tamely ramified extension of K . Then the unique extension of ∂ to L is of rational type (with u again as rational parameter).*

Proof. We reduce immediately to the case of a finite tamely ramified extension. The extension of ∂ to L is obtained from the isomorphism $\Omega_L^1 \cong L \otimes \Omega_K^1$. We need to prove that for each positive integer n and each $x \in L$, $|u^n \partial^n(x)/n!| \leq |x|$. We may consider the unramified extension and the totally tamely ramified extension separately.

Suppose first that L/K is unramified. Since every element of L equals an element of K times an element of \mathfrak{o}_L^\times , we need only check the inequality $|u^n \partial^n(x)/n!| \leq |x|$ for $x \in \mathfrak{o}_L^\times$. We do this by induction on n . Let $h(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \in \mathfrak{o}_K[T]$ be the minimal polynomial of x ; thus $h'(x) \in \mathfrak{o}_L^\times$. For the base case $n = 1$ of the induction, applying $u\partial$ to the equation $h(x) = 0$ gives

$$u\partial(x) = -\frac{u\partial(a_{d-1})x^{d-1} + \cdots + u\partial(a_0)}{h'(x)} \in \mathfrak{o}_L.$$

Assume the statement is proved for $n - 1$. Applying $u^n \partial^n / n!$ to the equation $h(x) = 0$ gives

$$\sum_{i=0}^d \sum_{\lambda_0 + \dots + \lambda_i = n} \frac{u^{\lambda_0} \partial^{\lambda_0}}{\lambda_0!}(a_i) \frac{u^{\lambda_1} \partial^{\lambda_1}}{\lambda_1!}(x) \cdots \frac{u^{\lambda_i} \partial^{\lambda_i}}{\lambda_i!}(x) = 0,$$

where $a_d = 1$ by convention. Each summand belongs to \mathfrak{o}_L by the induction hypothesis except for those in which $\lambda_j = n$ for some $j > 0$; those terms add up to $h'(x)u^n \partial^n(x)/n!$. Therefore $u^n \partial^n(x)/n! \in \mathfrak{o}_L$, completing the induction.

Now suppose that L/K is totally tamely ramified. We induct on $[L : K]$, which we may assume is greater than 1. Then we can find $d > 1$ and $x_0 \in \mathfrak{o}_L$ such that $|x_0^i| \notin |K^\times|$ for $i = 1, \dots, d - 1$. Choose an element $y \in \mathfrak{o}_K$ with $|y - x_0^d| < |x_0^d|$. By Hensel's lemma, y has a d -th root z in L . Let K' be the completion of $K(t)$ for the $|y|^{1/d}$ -Gauss norm, and extend ∂ to K' by setting $\partial(t) = 0$. The residue field of K' is $k(y/t^d)$. Put $L' = K' \otimes_K K(z)$; then $L' = K'(z) = K'(z/t)$. Now z/t is a d -th root of the quantity $y/t^d \in \mathfrak{o}_{K'}$, whose image in the residue field has no i -th root for any $i > 1$ dividing d . Hence L'/K' is unramified, so by the previous paragraph, ∂ extends to L' and is of rational type with respect to u . We may then read off the same conclusion for $K(z)$; applying the induction hypothesis to $L/K(z)$ yields the claim. \square

Hypothesis 1.4.6. For the rest of this subsection, we assume that K is a complete nonarchimedean field of characteristic zero and residual characteristic p , equipped with a differential operator ∂ of rational type with respect to the rational parameter u . We also assume $p > 0$ unless otherwise specified.

Construction 1.4.7. If K contains a primitive p -th root of unity ζ_p , we may define an action of the group $\mathbb{Z}/p\mathbb{Z}$ on K using ∂ -Taylor series:

$$x^{(i)} = \mathbb{T}(x; \partial, (\zeta_p^i - 1)u), \quad (i \in \mathbb{Z}/p\mathbb{Z}, x \in K).$$

It is clear that $|x^{(i)}| = |x|$ for $i \in \mathbb{Z}/p\mathbb{Z}$. Let $K^{(\partial)}$ be the fixed subfield of K under this action; in particular, $u^p \in K^{(\partial)}$. By simple Galois theory, K is a Galois extension of $K^{(\partial)}$ generated by u with Galois group $\mathbb{Z}/p\mathbb{Z}$. Moreover, $K^{(\partial)}$ is stable under the action of $u\partial$ because $(u\partial x)^{(i)} = u\partial(x^{(i)})$ for $x \in K$. (If K does not contain a primitive p -th root of unity, we may still define $K^{(\partial)}$ using Galois descent.)

We call the inclusion $\varphi^{(\partial)*} : K^{(\partial)} \hookrightarrow K$ the ∂ -Frobenius morphism. We view $K^{(\partial)}$ as being equipped with the derivation $\partial' = \partial/(pu^{p-1})$; we will see below (Lemma 1.4.9) that ∂' is of rational type with parameter u^p .

It is worth pointing out that $K^{(\partial)}$ depends on the choice of the rational parameter u , not just the derivation ∂ .

Occasionally, we use $K^{(\partial, n)}$ to denote the subfield of K obtained by applying the above construction n times; if K contains a primitive p^n -th root of unity, this is the same as the fixed field for the natural action of $\mathbb{Z}/p^n\mathbb{Z}$ on K .

Lemma 1.4.8. *We have $|\partial'|_{K^{(\partial)}} = |u|^{-p}$.*

Proof. We may assume that K contains a primitive p -th root of unity ζ_p . We need only show that $u^p \partial'$ preserves $\mathfrak{o}_{K^{(\partial)}}$. For any $x \in \mathfrak{o}_{K^{(\partial)}}$, we have

$$x = \frac{1}{p}(x + x^{(1)} + \cdots + x^{(p-1)}) = \frac{1}{p} \sum_{n=0}^{\infty} \frac{\partial^n(x)}{n!} u^n \sum_{i=0}^{p-1} (\zeta_p^i - 1)^n.$$

Applying $u^p \partial' = u \partial / p$ gives

$$\begin{aligned} u^p \partial'(x) &= \frac{u}{p^2} \sum_{n=0}^{\infty} \left(\frac{\partial^{n+1}(x)}{n!} u^n \sum_{i=0}^{p-1} (\zeta_p^i - 1)^n + \frac{\partial^n(x)}{(n-1)!} u^{n-1} \sum_{i=0}^{p-1} (\zeta_p^i - 1)^n \right) \\ &= \frac{u}{p^2} \sum_{n=0}^{\infty} \frac{\partial^{n+1}(x)}{n!} u^n \sum_{i=0}^{p-1} (\zeta_p^i - 1)^n \zeta_p^i. \end{aligned} \quad (1.4.8.1)$$

The sum $\sum_{i=0}^{p-1} (\zeta_p^i - 1)^n \zeta_p^i$ equals 0 for $n = 0, \dots, p-2$; it equals p for $n = p-1$; and it is a multiple of p^2 for any $n \geq p$ (because the quantity belongs both to \mathbb{Z} and to the ideal $(\zeta_p - 1)^p$ in $\mathbb{Z}[\zeta_p]$). Hence by (1.4.8.1), $u^p \partial'(x)$ equals $u^p \partial^p(x)/p!$ plus an element of \mathfrak{o}_K , yielding $u^p \partial'(x) \in \mathfrak{o}_K \cap K^{(\partial)} = \mathfrak{o}_{K^{(\partial)}}$. \square

Lemma 1.4.9. *The differential operator ∂' on $K^{(\partial)}$ is of rational type, with parameter u^p .*

Proof. Write

$$\begin{aligned} \frac{u^{pn} \partial'^n}{n!}(x) &= \frac{(u^p \partial')(u^p \partial' - 1) \cdots (u^p \partial' - (n-1))}{n!}(x) \\ &= \frac{(u \partial)(u \partial - p) \cdots (u \partial - (n-1)p)}{n! \cdot p^n}(x) \end{aligned}$$

As a corollary of Lemma 1.4.8, for any element $x \in K^{(\partial)}$ and $i \in \mathbb{Z} \setminus p\mathbb{Z}$, $|(u \partial - i)(x)| = |x|$. Since $u \partial$ fixes $K^{(\partial)}$, applying differential operators $u \partial - i$ for $i \in \mathbb{Z} \setminus p\mathbb{Z}$ to the result will not change the norm, so

$$\left| \frac{u^{pn} \partial'^n}{n!}(x) \right| = \left| \frac{(u \partial)(u \partial - 1) \cdots (u \partial - (np-1))}{n! \cdot p^n}(x) \right| = \left| \frac{u^{np} \partial^{np}}{(np)!}(x) \right|.$$

The statement follows. \square

Definition 1.4.10. Given a ∂' -differential module V' over $K^{(\partial)}$, we may view $\varphi^{(\partial)*} V' = V' \otimes_{K^{(\partial)}} K$ as a ∂ -differential module over K by setting

$$\partial(\mathbf{v}' \otimes x) = pu^{p-1} \partial'(\mathbf{v}') \otimes x + \mathbf{v}' \otimes \partial(x) \quad (\mathbf{v}' \in V', x \in K).$$

Lemma 1.4.11. *Let V' be a ∂' -differential module over $K^{(\partial)}$. Then*

$$IR_{\partial}(\varphi^{(\partial)*} V') \geq \min\{IR_{\partial'}(V')^{1/p}, p IR_{\partial'}(V')\}.$$

Proof. This is essentially [11, Lemma 10.3.2]. Consider the diagram

$$\begin{array}{ccc} K^{(\partial)} & \xrightarrow{f_{\text{gen}}'^*} & K^{(\partial)} \llbracket T'/u^p \rrbracket_0 \\ \downarrow \varphi^{(\partial)*} & & \downarrow \tilde{\varphi}^{(\partial)*} \\ K & \xrightarrow{f_{\text{gen}}^*} & K \llbracket T/u \rrbracket_0 \end{array}$$

where $\tilde{\varphi}^{(\partial)*}$ is a $K^{(\partial)}$ -homomorphism extending $\varphi^{(\partial)*}$ by $\tilde{\varphi}^{(\partial)*}(T') = (u + T)^p - u^p$. The diagram commutes because formally

$$\begin{aligned} (\tilde{\varphi}^{(\partial)*} \circ f_{\text{gen}}'^*)(x) &= \tilde{\varphi}^{(\partial)*} \left(\sum_{n=0}^{\infty} \binom{u^p \partial'}{n} (x) \left(\frac{T'}{u^p} \right)^n \right) = \sum_{n=0}^{\infty} \binom{u \partial/p}{n} (\varphi^{(\partial)*}(x)) \left(\left(1 + \frac{T}{u} \right)^p - 1 \right)^n \\ &= \left(1 + \left(1 + \frac{T}{u} \right)^p - 1 \right)^{u \partial/p} (\varphi^{(\partial)*}(x)) = \left(\left(1 + \frac{T}{u} \right)^p \right)^{u \partial/p} (\varphi^{(\partial)*}(x)) \\ &= \left(1 + \frac{T}{u} \right)^{u \partial} (\varphi^{(\partial)*}(x)) = \sum_{n=0}^{\infty} \binom{u \partial}{n} (\varphi^{(\partial)*}(x)) \left(\frac{T}{u} \right)^n = (f_{\text{gen}}^* \circ \varphi^{(\partial)*})(x). \end{aligned}$$

For $x \in K^{(\partial)}$, all of the series in this formal equation converge, and we obtain correct equalities.

For $r' \in [0, 1)$, set $r = \min\{(r')^{1/p}, pr'\}$, or equivalently, $r' = \max\{r^p, p^{-1}r\}$. By Proposition 1.2.14,

$$\begin{aligned} R_{\partial}(V') &\geq r'|u|^p \\ \Leftrightarrow f_{\text{gen}}'^* V' &\text{ is a trivial } \partial_{T'}\text{-differential module over } A_{K^{(\partial)}}[0, r'|u|^p] \\ \Rightarrow \tilde{\varphi}^{(\partial)*} f_{\text{gen}}'^* V' &= f_{\text{gen}}^* \varphi^{(\partial)*} V' \text{ is a trivial } \partial_T\text{-differential module over } A_K[0, r|u|] \\ \Leftrightarrow R_{\partial}(\varphi^{(\partial)*} V') &\geq r|u|. \end{aligned}$$

where the second implication is a direct corollary of the lemma below. The statement follows. \square

Lemma 1.4.12. [11, Lemma 10.2.2] *Let K be a nonarchimedean field. For $u, T \in K$ and $r \in (0, 1)$, if $|u - T| < r|u|$, then*

$$|u^p - T^p| \leq \max\{r^p|u|^p, p^{-1}r|u|^p\}.$$

Definition 1.4.13. For a ∂ -differential module V over K , define the ∂ -Frobenius descendant of V as the $K^{(\partial)}$ -module $\varphi_*^{(\partial)} V$ obtained from V by restriction along $\varphi^{(\partial)*} : K^{(\partial)} \rightarrow K$, viewed as a ∂' -differential module over $K^{(\partial)}$ with differential $\partial' = \frac{1}{pu^{p-1}}\partial$. Note that this operation commutes with duals.

Definition 1.4.14. For $n = 0, \dots, p-1$, let $W_n^{(\partial)}$ be the ∂' -differential module over $K^{(\partial)}$ with one generator \mathbf{v} , such that

$$\partial'(\mathbf{v}) = \frac{n}{p} u^{-p} \mathbf{v}.$$

From the Newton polynomial associated to \mathbf{v} , we read off $IR_{\partial'}(W_n^{(\partial)}) = p^{-p/(p-1)}$ for $n \neq 0$. (One may view the generator \mathbf{v} as a proxy for u^n .)

Lemma 1.4.15. *We have the following relations between ∂ -Frobenius pullbacks and ∂ -Frobenius descendants.*

(a) *For V a ∂ -differential module over K , there are canonical isomorphisms*

$$\iota_n : (\varphi_*^{(\partial)} V) \otimes W_n^{(\partial)} \simeq \varphi_*^{(\partial)} V \quad (n = 0, \dots, p-1).$$

(b) *For V a ∂ -differential module over K , a submodule U of $\varphi_*^{(\partial)} V$ is itself the ∂ -Frobenius descendant of a submodule of V if and only if $\iota_n(U \otimes W_n^{(\partial)}) = U$ for $n = 0, \dots, p-1$.*

(c) *For V a ∂ -differential module over K , there is a canonical isomorphism*

$$\varphi^{(\partial)*} \varphi_*^{(\partial)} V \simeq V^{\oplus p}.$$

(d) *For V' a ∂' -differential module over $K^{(\partial)}$, there is a canonical isomorphism*

$$\varphi_*^{(\partial)} \varphi^{(\partial)*} V' \simeq \bigoplus_{n=0}^{p-1} (V' \otimes W_n^{(\partial)}).$$

(e) *For V_1, V_2 ∂ -differential modules over K , there is a canonical isomorphism*

$$\varphi_*^{(\partial)} V_1 \otimes \varphi_*^{(\partial)} V_2 \simeq \bigoplus_{n=0}^{p-1} W_n^{(\partial)} \otimes \varphi_*^{(\partial)} (V_1 \otimes V_2).$$

(f) *For V a ∂ -differential module over K , there are canonical bijections*

$$H_{\partial}^i(V) \simeq H_{\partial'}^i(\varphi_*^{(\partial)} V) \quad (i = 0, 1).$$

Proof. Straightforward. □

Definition 1.4.16. Let V be a ∂ -differential module over K such that $IR_{\partial}(V) > p^{-1/(p-1)}$. A ∂ -Frobenius antecedent of V is a ∂' -differential module V' over $K^{(\partial)}$ such that $V \simeq \varphi^{(\partial)*} V'$ and $IR_{\partial'}(V') > p^{-p/(p-1)}$.

Proposition 1.4.17 (Christol-Dwork). *Let V be a ∂ -differential module over K such that $IR_{\partial}(V) > p^{-1/(p-1)}$. Then there exists a unique ∂ -Frobenius antecedent V' of V . Moreover, $IR_{\partial'}(V') = IR_{\partial}(V)^p$.*

Proof. As in [11, Theorem 10.4.2]. □

Remark 1.4.18. As in [11, Theorem 10.4.4], one can form a version of Proposition 1.4.17 for differential modules over discs and annuli.

Theorem 1.4.19. *Let V be a ∂ -differential module over K . Then*

$$\mathfrak{IR}_{\partial'}(\varphi_*^{(\partial)}V) = \bigcup_{r \in \mathfrak{IR}_{\partial}(V)} \begin{cases} \{r^p, p^{-p/(p-1)} (p-1 \text{ times})\} & r > p^{-1/(p-1)} \\ \{p^{-1}r (p \text{ times})\} & r \leq p^{-1/(p-1)}. \end{cases}$$

In particular, $\mathfrak{IR}_{\partial'}(\varphi_^{(\partial)}V) = \min\{p^{-1}\mathfrak{IR}_{\partial}(V), p^{-p/(p-1)}\}$.*

Proof. The proof is identical to that of [11, Theorem 10.5.1]. \square

Corollary 1.4.20. *Let V' be a ∂' -differential module over $K^{(\partial)}$ such that $\mathfrak{IR}_{\partial'}(V') \neq p^{-p/(p-1)}$. Then $\mathfrak{IR}_{\partial}(\varphi^{(\partial)*}V') = \min\{\mathfrak{IR}_{\partial'}(V')^{1/p}, p\mathfrak{IR}_{\partial'}(V')\}$.*

Proof. In case $\mathfrak{IR}_{\partial'}(V') > p^{-p/(p-1)}$, this holds by [11, Corollary 10.4.3]. Otherwise, by Lemma 1.4.15(d), $\varphi_*^{(\partial)}\varphi^{(\partial)*}V' \cong \bigoplus_{m=0}^{p-1}(V' \otimes W_m^{(\partial)})$ and $\mathfrak{IR}_{\partial'}(V' \otimes W_m^{(\partial)}) = \mathfrak{IR}_{\partial'}(V')$ since $\mathfrak{IR}_{\partial'}(V') < \mathfrak{IR}_{\partial'}(W_m^{(\partial)})$. Hence by Theorem 1.4.19,

$$\mathfrak{IR}_{\partial'}(V') = \mathfrak{IR}_{\partial'}(\varphi_*^{(\partial)}\varphi^{(\partial)*}V') = \min\{p^{-1}\mathfrak{IR}_{\partial}(\varphi^{(\partial)*}V'), p^{-p/(p-1)}\}.$$

We get a contradiction if the right side equals $p^{-p/(p-1)}$, so we must have $\mathfrak{IR}_{\partial'}(V') = p^{-1}\mathfrak{IR}_{\partial}(\varphi^{(\partial)*}V') \leq p^{-p/(p-1)}$, proving the claim. \square

For the following theorem, we do not assume $p > 0$.

Theorem 1.4.21. *Let V be a ∂ -differential module over K . Then there exists a decomposition*

$$V = \bigoplus_{r \in (0,1]} V_r,$$

where every subquotient of V_r has pure intrinsic ∂ -radii r . Moreover, if $p = 0$, then $r^{\dim V_r} \in |K^\times|$; if $p > 0$, then for any nonnegative integer h , we have

$$r < p^{-p^{-h}/(p-1)} \implies r^{\dim V_r} \in |(K^{(\partial,h)})^\times|^{p^{-h}}.$$

Proof. The proof is similar to those of [11, Theorem 10.6.2] and [11, Theorem 10.7.1]. \square

Remark 1.4.22. In the case when K is the completion of $K_0(u)$ with respect to the η -Gauss norm, $K^{(\partial,h)}$ is the completion of $K_0(u^{p^h})$ with respect to the η^{p^h} -Gauss norm. We deduce thus from Theorem 1.4.21 that $r^{\dim V_r} \in |K_0^\times|^{p^{-h}}\eta^{\mathbb{Z}}$.

Remark 1.4.23. Let K' be a complete extension of K equipped with an extension of ∂ which is again of rational type with parameter u . Then the intrinsic radii of a ∂ -differential module over K are the same as that of its base extension to K' : namely, this is clear from Remark 1.3.5 for those radii less than ω , but we can reduce to this case using Theorem 1.4.19.

1.5 Multiple derivations

In this subsection, we introduce differential fields of higher order.

Definition 1.5.1. Let K denote a differential ring of order m , i.e., a ring K equipped with m commuting derivations $\partial_1, \dots, \partial_m$. For $j \in J = \{1, \dots, m\}$, a ∂_j -differential module is a finite projective K -module V equipped with the action of ∂_j . In other words, we view K as a differential ring of order 1 by forgetting the derivations other than ∂_j . A $(\partial_1, \dots, \partial_m)$ -differential module (or ∂_j -differential module, or simply a differential module) is a finite projective K -module V equipped with commuting actions of $\partial_1, \dots, \partial_m$. We may apply the results above by singling out one of $\partial_1, \dots, \partial_m$.

Definition 1.5.2. Let K be a complete nonarchimedean differential field of order m and characteristic zero, and let V be a nonzero $(\partial_1, \dots, \partial_m)$ -differential module over K . Define the *intrinsic generic radius of convergence*, or for short the *intrinsic radius*, of V to be

$$IR(V) = \min_{j \in J} \{IR_{\partial_j}(V)\} = \min_{j \in J} \{|\partial_j|_{\text{sp}, K} / |\partial_j|_{\text{sp}, V}\}.$$

For $j \in J$, we say ∂_j is *dominant* for V if $IR_{\partial_j}(V) = IR(V)$. We define the *intrinsic subsidiary radii* $\mathfrak{IR}(V) = \{IR(V; 1), \dots, IR(V; \dim V)\}$ by collecting and ordering intrinsic radii from Jordan-Hölder factors, as in Definition 1.2.8. We again say that V has *pure intrinsic radii* if the elements of $\mathfrak{IR}(V)$ are all equal to $IR(V)$.

Definition 1.5.3. Let K be a complete nonarchimedean differential field of order m and characteristic zero. We say that K is of *rational type* with respect to a set of parameters $\{u_j : j \in J\}$ if each ∂_j is of rational type with respect to u_j , and $\partial_i(u_j) = 0$ for $i \neq j$.

Remark 1.5.4. Set notation as in Definition 1.5.3. Let K' be the completion of $K(t)$ for the η -Gauss norm; then K' is again of rational type with respect to u_1, \dots, u_m, t .

Remark 1.5.5. Recall that if $p > 0$, we have a ∂_j -Frobenius $\varphi^{(\partial_j)*} : K^{(\partial_j)} \hookrightarrow K$ for $j \in J$. Since the elements $u_{J \setminus \{j\}}$ are killed by ∂_j , they are elements in $K^{(\partial_j)}$. Hence by Lemma 1.4.9, the differential operators $\partial_{J \setminus \{j\}}$ and ∂'_j are of rational type over $K^{(\partial_j)}$ with respect to the parameters $u_{J \setminus \{j\}}$ and u_j^p .

Theorem 1.5.6. *Let K be a complete nonarchimedean differential field of order m and characteristic zero, of rational type. Let V be a ∂_J -differential module over K . Then there exists a decomposition*

$$V = \bigoplus_{r \in (0, 1]} V_r,$$

where every subquotient of V_r has pure intrinsic radii r . Moreover, if $p = 0$, then $r^{\dim V_r} \in |K^\times|$; if $p > 0$, then

$$r < p^{-p^{-h}/(p-1)} \implies r^{\dim V_r} \in |K^\times|^{1/p^h}.$$

Proof. Since the ∂_j commute with each other, the theorem follows by applying Theorem 1.4.21 to each ∂_j and forming a common refinement of the resulting decompositions. \square

Definition 1.5.7. For l/k an extension of fields of characteristic $p > 0$, we say the extension is *separable* if $l \cap k^{p^{-1}} = k$. A p -*basis* of l over k is a set $B = \{u_j\}_{j \in J} \subset l$ such that the products $u_j^{e_j}$, where $e_j \in \{0, 1, \dots, p-1\}$ for all $j \in J$ and $e_j = 0$ for all but finitely many j , form a basis of the vector space l over kl^p . By a p -*basis* of l we mean a p -basis of l over l^p . (For more details, see [5, p. 565] or [6, Ch.0, §21].)

For an extension L/K of complete nonarchimedean fields with residue fields l, k of characteristic $p > 0$, with l/k separable, a p -*basis* of L over K will mean a set of elements $(u_J) \subset \mathfrak{o}_L^\times$ whose images $(\bar{u}_J) \subset l$ form a p -basis of l over k .

One important instance of Definition 1.5.3 is the following.

Situation 1.5.8. Let m be a nonnegative integer and $J = \{1, \dots, m\}$. Let F be a complete discrete valuation field of characteristic 0 with residue field κ of characteristic $p > 0$. Let K_1 be a complete extension of F with the same value group and residue field k_1 separable over κ . Assume K_1 has a finite p -basis (u_1, \dots, u_m) over F . Let F' be an extension of F complete for a (not necessarily discrete) nonarchimedean norm $|\cdot|$, with the same residue field κ . Let K_2 be the completion of $K_1 \otimes_F F'$. Let k be a (possibly infinite) separable algebraic extension of k_1 , and let K be the completion of the unramified extension of K_2 with residue field k .

Lemma 1.5.9. *In Situation 1.5.8, the natural projection $\Omega_K^1 \rightarrow \bigoplus_{j=1}^m K \cdot du_j$ gives derivations $(\partial_j = \partial_{u_j})_{j \in J}$ of rational type with respect to u_1, \dots, u_m .*

Proof. It is enough to check for K_1 : it is clear that the same conclusion then holds for K_2 , and then Lemma 1.4.5 implies the same conclusion for K . That is, we must check that \mathfrak{o}_{K_1} is stable under $\partial_j^n/n!$ for all nonnegative integers n and all $j \in J$. For each $n \in \mathbb{N}$, any element $x \in \mathfrak{o}_{K_1}$ can be written (not uniquely) as

$$x = \sum_{i=0}^{+\infty} \sum_{e_J=0}^{p^n-1} \alpha_{n,i,e_J}^{p^n} u_J^{e_J} \pi_F^i,$$

where $\alpha_{n,i,e_J} \in \mathfrak{o}_{K_1}^\times \cup \{0\}$. Then for any $j_0 \in J$,

$$\frac{\partial_{j_0}^n}{n!}(x) = \sum_{i=0}^{+\infty} \sum_{e_J=0}^{p^n-1} \sum_{\beta=0}^n \frac{\partial_{j_0}^\beta}{\beta!} \left(\alpha_{n,i,e_J}^{p^n} \right) \frac{\partial_{j_0}^{n-\beta}}{(n-\beta)!} (u_J^{e_J}) \pi_F^i \in \mathfrak{o}_{K_1}.$$

The lemma follows. □

Remark 1.5.10. Situation 1.5.8 includes the two options in [8, Hypothesis 2.1.3]. (Note that [8, Hypothesis 2.1.3(b)] should require that l/k be separable.) We will see later (Theorem 2.6.1) that the results in [8] carry over to differential fields of rational type.

2 Differential modules on one-dimensional spaces

Having considered differential modules over fields, we next consider differential modules on a disc or annulus over a differential field. This parallels [11, Chapters 11 and 12].

Hypothesis 2.0.1. Throughout this section, we assume that K is a complete (not necessarily discretely valued) nonarchimedean differential field of order m , characteristic zero, and residual characteristic p (not necessarily positive). We also assume K is of rational type.

Notation 2.0.2. Let $\partial_1, \dots, \partial_m$ denote the derivatives on K and let u_1, \dots, u_m denote a set of corresponding rational parameters. Let $J = \{1, \dots, m\}$. We reserve j and J for indexing derivations.

2.1 Setup

Notation 2.1.1. For $\eta > 0$, let F_η be the completion of $K(t)$ under the η -Gauss norm $|\cdot|_\eta$. Put $\partial_0 = \frac{d}{dt}$ on F_η ; by Remark 1.5.4, F_η is of rational type for the derivations ∂_{J^+} , where $J^+ = J \cup \{0\} = \{0, \dots, m\}$.

Remark 2.1.2. For $I \subseteq [0, +\infty)$ an interval and $j \in J^+$, we may refer to differential modules or ∂_j -differential modules over $A_K^1(I)$, meaning locally free coherent sheaves with the appropriate derivations. For $I = [\alpha, \beta]$ closed, these are just modules with appropriate derivations over the principal ideal domain $K\langle \alpha/t, t/\beta \rangle$; in particular, any ∂_j -differential module over a closed annulus is free by [11, Proposition 9.1.2].

Remark 2.1.3. For $I \subseteq [0, +\infty)$ an interval, and M a nonzero ∂_j -differential module over $A_K^1(I)$, it is unambiguous to refer to the intrinsic ∂_j -radius of convergence $IR_{\partial_j}(M \otimes F_\eta)$ of M at $|t| = \eta$.

The intrinsic radii are stable under tame base change.

Proposition 2.1.4. *Let n be a (possibly negative) nonzero integer (coprime to p if $p > 0$), and let $f_n^*: F_\eta \rightarrow F_{\eta^{1/n}}$ be the map $t \rightarrow t^n$. Then for any $j \in J^+$, and for any ∂_j -differential module V over F_η , $IR_{\partial_j}(V) = IR_{\partial_j}(f_n^*V)$ and hence $\mathfrak{IR}_{\partial_j}(V) = \mathfrak{IR}_{\partial_j}(f_n^*V)$.*

Proof. The proof for $j = 0$ is in [11, Proposition 9.7.6], and the proof for $j \in J$ is to apply Remark 1.2.7. \square

Remark 2.1.5. One may also consider off-centered tame base change, as in [11, Exercise 9.8].

2.2 Variation of subsidiary radii

In this subsection, we prove slightly weakened analogues of some results in [11, Chapter 11]. We begin by studying the variation of slopes of Newton polygons.

Notation 2.2.1. Let $P \in K\langle \alpha/t, t/\beta \rangle[T]$ be a polynomial of degree d . For $r \in [-\log \beta, -\log \alpha]$, let $\text{NP}_r(P)$ denote the Newton polygon of P under $|\cdot|_{e^{-r}}$.

Proposition 2.2.2. For $r \in [-\log \beta, -\log \alpha]$, let $f_1(P, r), \dots, f_d(P, r)$ be the slopes of $\text{NP}_r(P)$ in increasing order. For $i = 1, \dots, d$, put $F_i(P, r) = f_1(P, r) + \dots + f_i(P, r)$.

- (a) (Linearity) For $i = 1, \dots, d$, the functions $f_i(P, r)$ and $F_i(P, r)$ are continuous and piecewise affine in r .
- (b) (Integrality) If $i = d$ or $f_i(r_0) < f_{i+1}(r_0)$, then the slopes of $F_i(P, r)$ in some neighborhood of $r = r_0$ belong to \mathbb{Z} . Consequently, the slopes of each $f_i(P, r)$ and $F_i(P, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \dots \cup \frac{1}{d}\mathbb{Z}$.
- (c) (Monotonicity) Suppose that P is monic and $\alpha = 0$. For $i = 1, \dots, d$, the slopes of $F_i(P, r)$ are nonnegative.
- (d) (Concavity) Suppose that P is monic. For $i = 1, \dots, d$, the function $F_i(P, r)$ is concave.
- (e) (Truncation) For any fixed $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, the statements (a), (c), and (d) are also true if we replace $f_i(P, r)$ by $\min\{f_i(P, r), ar + b\}$ for all $i \in \{1, \dots, d\}$.

Proof. See [11, Theorem 11.2.1] and [11, Remark 11.2.4]. □

Lemma 2.2.3 (Lattice lemma). Put $R = K\langle t \rangle$, $\cup_{\alpha < 1} K\langle \alpha/t, t \rangle$, or $\cup_{\alpha < 1 < \beta} K\langle \alpha/t, t/\beta \rangle$, or (if K is discrete) $K[[t]]_0$ or $\cup_{\alpha < 1} K[[\alpha/t, t]]_0$ equipped with the norm $|\cdot|_1$. Let M be a finite free R -module of rank n , and let $|\cdot|_M$ be a norm on M compatible with R . Assume that either:

- (a) $c > 1$, and the value group of K is not discrete; or
- (b) $c \geq 1$, and the value groups of K and M coincide and are discrete.

Then there exists a basis of M defining a supremum norm $|\cdot|'_M$ for which $c^{-1}|m|_M \leq |m|'_M \leq c|m|_M$ for $m \in M$.

Proof. Let F be the completion of $\text{Frac } R$ under $|\cdot|_1$. By [11, Lemma 1.3.7], we can construct a basis of $M \otimes F$ defining a supremum norm $|\cdot|'_M$ for which $c^{-1}|m|_M \leq |m|'_M \leq c|m|_M$ for $m \in M$. If $R = K\langle t \rangle$, or K is discrete and $R = K[[t]]_0$, then [11, Lemma 8.6.1] gives a basis of M defining the same supremum norm $|\cdot|'_M$. If $R = \cup_{\alpha < 1} K\langle \alpha/t, t \rangle$ or $\cup_{\alpha < 1 < \beta} K\langle \alpha/t, t/\beta \rangle$, then [11, Lemma 8.6.1] gives a basis of $K\langle 1/t, t \rangle$ defining $|\cdot|'_M$. However, we can approximate that basis arbitrarily closely with a basis of M itself, because R is dense in $K\langle 1/t, t \rangle$ under $|\cdot|_1$, and any element of R with an inverse in $K\langle 1/t, t \rangle$ also has an inverse in R . Any sufficiently good approximation will define the same supremum norm. If K is discrete and $R = \cup_{\alpha < 1} K[[\alpha/t, t]]_0$, then R itself is a field, so we can approximate a basis of $M \otimes F$ with a basis of M defining the same supremum norm. □

Notation 2.2.4. Fix $j \in J^+$. Let M be a ∂_j -differential module of rank d over $K\langle \alpha/t, t/\beta \rangle$. For $r \in [-\log \beta, -\log \alpha]$ and $i \in \{1, \dots, d\}$, define

$$f_i^{(j)}(M, r) = -\log R_{\partial_j}(M \otimes F_{e^{-r}}; i), \quad F_i^{(j)}(M, r) = f_1^{(j)}(M, r) + \dots + f_i^{(j)}(M, r).$$

Theorem 2.2.5. [11, Theorem 11.3.2] Let M be a ∂_0 -differential module of rank d over $K\langle\alpha/t, t/\beta\rangle$.

- (a) (Linearity) For $i = 1, \dots, d$, the functions $f_i^{(0)}(M, r)$ and $F_i^{(0)}(M, r)$ are continuous and piecewise affine.
- (b) (Integrality) If $i = d$ or $f_i^{(0)}(M, r_0) > f_{i+1}^{(0)}(M, r_0)$, then the slopes of $F_i^{(0)}(M, r)$ in some neighborhood of r_0 belong to \mathbb{Z} . Consequently, the slopes of each $f_i^{(0)}(M, r)$ and $F_i^{(0)}(M, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \dots \cup \frac{1}{d}\mathbb{Z}$.
- (c) (Monotonicity) Suppose that $\alpha = 0$. For any point r_0 where $f_i^{(0)}(M, r_0) > r_0$, the slopes of $F_i^{(0)}(M, r)$ are nonpositive in some neighborhood of r_0 . Also, $f_i^{(0)}(M, r_0) = r_0$ for r_0 sufficiently large.
- (d) (Convexity) For $i = 1, \dots, d$, the function $F_i^{(0)}(M, r)$ is convex.

We have a similar but slightly weaker result for ∂_j -differential modules when $j \in J$.

Theorem 2.2.6. Fix $j \in J$. Let M be a ∂_j -differential module of rank d over $K\langle\alpha/t, t/\beta\rangle$.

- (a) (Linearity) For $i = 1, \dots, d$, the functions $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ are continuous. They are piecewise affine in the locus where $f_i^{(j)}(M, r) > -\log |u_j|$; if $p = 0$, they are in fact piecewise affine everywhere.
- (b) (Weak integrality)
 - (i) Suppose $p = 0$. If $i = d$ or $f_i^{(j)}(M, r_0) > f_{i+1}^{(j)}(M, r_0)$, then the slopes of $F_i^{(j)}(M, r)$ in some neighborhood of r_0 belong to \mathbb{Z} . Consequently, the slopes of each $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ at $r = r_0$ belong to $\frac{1}{1}\mathbb{Z} \cup \dots \cup \frac{1}{d}\mathbb{Z}$.
 - (ii) Suppose $p > 0$. If $i = d$ or $f_i^{(j)}(M, r_0) > f_{i+1}^{(j)}(M, r_0)$, and $f_i^{(j)}(M, r_0) > \frac{1}{p^n(p-1)} \log p - \log |u_j|$ for some $n \in \mathbb{Z}_{\geq 0}$, then the slopes of $F_i^{(j)}(M, r)$ in some neighborhood of r_0 belong to $\frac{1}{p^n}\mathbb{Z}$. Consequently, if $f_i^{(j)}(M, r_0) > \frac{1}{p^n(p-1)} \log p - \log |u_j|$ for some $n \in \mathbb{Z}_{\geq 0}$, the slopes of each $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ at $r = r_0$ belong to $\frac{1}{p^n}\mathbb{Z} \cup \dots \cup \frac{1}{p^n d}\mathbb{Z}$.
- (c) (Monotonicity) Suppose that $\alpha = 0$. For $i = 1, \dots, d$, the slopes of $F_i^{(j)}(M, r)$ are nonpositive.
- (d) (Convexity) For $i = 1, \dots, d$, the function $F_i^{(j)}(M, r)$ is convex.

Proof. We prove the theorem analogously to [11, Theorem 11.3.2]. First of all, as in Remark 1.5.4, we may replace K by the completion of $K(x)$ with respect to the $|u_j|$ -Gauss norm. We may then replace u_j by u_j/x to reduce to the case $|u_j| = 1$.

We first show that the statements are true for $\tilde{f}_i^{(j)}(M, r) = \max\{f_i^{(j)}(M, r), \epsilon\}$ with $\epsilon > -\log \omega$ and $\tilde{F}_i^{(j)}(M, r) = \tilde{f}_1^{(j)}(M, r) + \cdots + \tilde{f}_i^{(j)}(M, r)$. Let $F = \text{Frac } K\langle \alpha/t, t/\beta \rangle$. Choose a cyclic vector for $M \otimes F$ to obtain an isomorphism $M \otimes F \cong F\{T\}/F\{T\}P$ for some monic twisted polynomial P over F . We may then apply Proposition 2.2.2 and Remark 1.3.5 to deduce (a) and (b), provided we omit the last assertion in (a) (in case $p = 0$); for that, see below.

For (c) and (d), it suffices to work in a neighborhood of some r_0 . Again by Remark 1.5.4, there is no harm in enlarging K so that $e^{-r_0} \in |K^\times|$. We may reduce to the case $r_0 = 0$ by replacing t by λt for some $\lambda \in K^\times$ with $|\lambda| = e^{-r_0}$. We then argue as in [11, Lemma 11.5.1] and deduce (c) and (d) from Proposition 2.2.2, as follows. We may further enlarge K to include $\lambda_1, \dots, \lambda_n \in \ker(\partial_j)$ such that

$$-\log |\lambda_j| = \min \{-\log \omega - f_j(M, 0), 0\} \quad (j = 1, \dots, d).$$

Let B_0 be the basis of $M \otimes F_1$ given by

$$\lambda_{d-1}^{-1} \cdots \lambda_{d-j}^{-1} T^j \quad (j = 0, \dots, d-1).$$

Let N_0 be the characteristic polynomial of the matrix of action of ∂_j on B_0 . Let μ_1, \dots, μ_n be the eigenvalues of N_0 , labeled so that $|\mu_1| \geq \cdots \geq |\mu_n|$. By [11, Proposition 4.3.10], we have $\max\{|\mu_j|, 1\} = \max\{\omega e^{f_j(M, 0)}, 1\}$ for $j = 1, \dots, d$. By Lemma 2.2.3, for each $c > 1$, we may construct a basis B_c of M such that the supremum norms $|\cdot|_0, |\cdot|_c$ defined by B_0, B_c satisfy $c^{-1}|\cdot|_c \leq |\cdot|_0 \leq c|\cdot|_c$. Let N_c be the matrix of action of ∂_j on B_c . For $c > 1$ sufficiently small, [11, Theorem 6.7.4] implies that for r close to 0, the visible spectrum of $M \otimes F_{e^{-r}}$ is the multiset of those norms of eigenvalues of the characteristic polynomial of N_c which exceed 1. (Here the *visible spectrum* of $M \otimes F_{e^{-r}}$ is defined as in [11, Definition 6.5.1], i.e., those spectral norms of subquotients of $M \otimes F_{e^{-r}}$ which exceed 1, with appropriate multiplicities.) We may then deduce (c) and (d) from Proposition 2.2.2(c) and (d).

We next relax the truncation condition that we have imposed; we may assume $p > 0$ as otherwise there is nothing to check. For each nonnegative integer n , we prove the claim for $\tilde{f}_i^{(j)}(M, r) = \max\{f_i^{(j)}(M, r), \epsilon\}$ and $\tilde{F}_i^{(j)}(M, r) = \tilde{f}_1^{(j)}(M, r) + \cdots + \tilde{f}_i^{(j)}(M, r)$ with $\epsilon \in \left(\frac{1}{p^n(p-1)} \log p, \frac{1}{p^{n-1}(p-1)} \log p\right]$, by induction on n ; the base case $n = 0$ is proved above. As above, we may reduce to the case $r_0 = 0$.

Consider the ∂_j -Frobenius $\varphi^{(\partial_j)*} : F_{e^{-r}}^{(\partial_j)} \hookrightarrow F_{e^{-r}}$. Put $g_i^{(j)}(r) = -\log R_{\partial_j}(\varphi_*^{(\partial_j)} M \otimes F_{e^{-r}}^{(\partial_j)}; i)$ and $\tilde{g}_i^{(j)}(r) = \max\{g_i^{(j)}(r), p\epsilon\}$ for $i = 1, \dots, pd$. By Theorem 1.4.19, the list $\{g_1^{(j)}(r), \dots, g_{pd}^{(j)}(r)\}$ consists of

$$\bigcup_{i=1}^d \left\{ \begin{array}{ll} \{p f_i^{(j)}(M, r), \frac{p}{p-1} \log p \text{ (} p-1 \text{ times)}\} & f_i^{(j)}(M, r) \leq \frac{1}{p-1} \log p \\ \{\log p + f_i^{(j)}(M, r) \text{ (} p \text{ times)}\} & f_i^{(j)}(M, r) \geq \frac{1}{p-1} \log p. \end{array} \right.$$

Thus, the list $\tilde{g}_1^{(j)}(r), \dots, \tilde{g}_{pd}^{(j)}(r)$ consists of

$$\bigcup_{i=1}^d \left\{ \begin{array}{ll} \{p \tilde{f}_i^{(j)}(M, r), \frac{p}{p-1} \log p \text{ (} p-1 \text{ times)}\} & f_i^{(j)}(M, r) \leq \frac{1}{p-1} \log p \\ \{\log p + \tilde{f}_i^{(j)}(M, r) \text{ (} p \text{ times)}\} & f_i^{(j)}(M, r) \geq \frac{1}{p-1} \log p. \end{array} \right.$$

We may thus deduce (a) and (b) directly from the induction hypothesis. We similarly deduce (d) as in [11, Lemma 11.6.1], except that we are considering $\tilde{g}_i^{(j)}(r)$ but not $\tilde{g}_i^{(j)}(pr)$; this explains the weakened integrality result. (See also Remark 1.4.22.) Also, we can luckily deduce (c) directly, because $\varphi^{(\partial_j)^*}$ does not introduce a singularity on $A_K^1[0, \beta]$; by contrast, in the proof of [11, Theorem 11.3.2], one must switch to an off-centered Frobenius to avoid a singularity at $t = 0$.

We deduce that (a)-(d) hold for $\tilde{f}_i^{(j)}(M, r) = \max\{f_i^{(j)}(M, r), \epsilon\}$ and $\tilde{F}_i^{(j)}(M, r) = \tilde{f}_1^{(j)}(M, r) + \dots + \tilde{f}_i^{(j)}(M, r)$ with $\epsilon > 0$. The desired results hold by taking $\epsilon \rightarrow 0^+$.

This completes the proof except that if $p = 0$, we must still prove piecewise affinity everywhere. In this case, the integrality of (b) is not burdened with an extra denominator of p^n , so we may repeat the argument from [11, Lemma 11.6.3]; see Step 3 of Theorem 2.4.4 for essentially the same argument. \square

Example 2.2.7. When $j \in J$, we do not expect an integrality result as in the $j = 0$ case; see Remark 1.4.22. One can easily generate an example in which the strong integrality statement for ∂_j fails, as follows. Suppose $p > 0$, $\alpha \in (p^{-1/(p-1)}, 1)$, and $|u_j| = 1$. We take the rank one ∂_j -differential module M over $K\langle \alpha/t, t \rangle$ generated by \mathbf{v} with $\partial_j(\mathbf{v}) = t^{-1}\mathbf{v}$. Thus, $f_1^{(j)}(M, r) = r$ for $r \in [0, -\log \alpha]$. By Corollary 1.4.20, $f_1^{(j)}(\varphi^{(\partial_j)^*}M, r) = \frac{r}{p}$.

Remark 2.2.8. Besides the weakening of the integrality condition, there are some other aspects in which Theorem 2.2.6 is weaker than its counterpart [11, Theorem 11.3.2] if $p > 0$. For one, the latter includes a subharmonicity assertion, which refers to the algebraic closure of the residue field of K . It is awkward to add a subharmonicity assertion here because the residue field of K is crucially imperfect, so that it can admit a nontrivial p -basis. (By contrast, if $p = 0$, we can achieve a subharmonicity result; see Theorem 2.7.6.) For another, Theorem 2.2.6(a) does not apply in a neighborhood of a point r_0 at which $f_i^{(j)}(M, r_0) = -\log |u_j|$. The argument in [11, Lemma 11.6.3] does not extend to this case because the weak integrality result does not give a lower bound on slopes. On the other hand, we do not have a counterexample against the claim that $f_i^{(j)}(M, r)$ is everywhere piecewise affine.

2.3 Decomposition by subsidiary radii

In this subsection, we prove some decomposition theorems over annuli and discs, as in [11, Chapter 12]. We start by a technical lemma, copied from [12, Lemma 1.2.7].

Lemma 2.3.1. *Let*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

be a commuting diagram of inclusions of integral domains, such that the intersection $S \cap T$ within U is equal to R . Let M be a finite locally free R -module. Then the intersection of $M \otimes_R S$ and $M \otimes_R T$ within $M \otimes_R U$ is equal to M .

Proof. Choose $\mathbf{e}_1, \dots, \mathbf{e}_n \in M$ which form a basis of $M \otimes_R (\text{Frac } R)$; then there exists $f \in R$ such that $fM \subseteq R\mathbf{e}_1 + \dots + R\mathbf{e}_n$. Given $\mathbf{v} \in M \otimes_R U$ which belongs to both $M \otimes_R S$ and $M \otimes_R T$, we can uniquely write $f\mathbf{v} = c_1\mathbf{e}_1 + \dots + c_n\mathbf{e}_n$ with $c_i \in U$. From the intersection property, we have $c_i \in R$ for $i = 1, \dots, n$, whence $f\mathbf{v} \in M$.

Since M is locally free, as we vary the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, the values of f obtained generate the unit ideal of R . We thus have $\mathbf{v} \in M$, as desired. \square

Lemma 2.3.2. *Retain notation as in Lemma 2.3.1. Then any direct sum decompositions of $M \otimes_R S$ and $M \otimes_R T$ which agree on $M \otimes_R U$ are induced by a unique direct sum decomposition of M .*

Proof. Apply Lemma 2.3.2 to the idempotents in $M^\vee \otimes M$ giving the projections onto the factors in the decompositions. \square

Lemma 2.3.3. *Given $\alpha < \beta$ and $x \in K\{\{\alpha/t, t/\beta\}\}$ such that the function $r \mapsto \log |x|_{e^{-r}}$ is affine for $r \in (-\log \beta, -\log \alpha)$, then x is a unit in $K\{\{\alpha/t, t/\beta\}\}$.*

Proof. The condition is equivalent to saying that the Newton polygon of x does not have any slopes in $(-\log \beta, -\log \alpha)$. This immediately implies the claim. \square

Lemma 2.3.4. *Let $P = \sum_i P_i T^i$ and $Q = \sum_i Q_i T^i$ be polynomials over $K\langle \alpha/t, t/\beta \rangle$ satisfying the following conditions.*

- (a) *We have $|P - 1|_\gamma < 1$ for all $\gamma \in [\alpha, \beta]$.*
- (b) *For $d = \deg(Q)$, Q_d is a unit and $|Q|_\gamma = |Q_d|_\gamma$ for all $\gamma \in [\alpha, \beta]$.*

Then P and Q generate the unit ideal in $K\langle \alpha/t, t/\beta \rangle[T]$.

Proof. We may assume without loss of generality that $Q_d = 1$. The hypothesis that $|Q|_\gamma = |Q_d|_\gamma$ for all $\gamma \in [\alpha, \beta]$ implies that if S is the remainder upon dividing R by Q , then $|S|_\gamma \leq |R|_\gamma$ for all $\gamma \in [\alpha, \beta]$ (compare [11, Lemma 2.3.1]). If we then let S_i denote the remainder upon dividing $(1 - P)^i$ by Q , the series $\sum_{i=0}^\infty S_i$ converges in $K\langle \alpha/t, t/\beta \rangle[T]$ (since the degrees of the S_i are bounded by $d - 1$) and its limit S satisfies $PS \equiv 1 \pmod{Q}$. \square

Theorem 2.3.5. *Fix $j \in J^+$. Let M be a ∂_j -differential module of rank d on $A_K^1(\alpha, \beta)$. Suppose that the following conditions hold for some $i \in \{1, \dots, d - 1\}$.*

- (a) *The function $F_i^{(j)}(M, r)$ is affine for $-\log \beta < r < -\log \alpha$.*
- (b) *We have $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $-\log \beta < r < -\log \alpha$.*

Then M admits a unique direct sum decomposition separating the first i subsidiary ∂_j -radii of $M \otimes F_\eta$ for any $\eta \in (\alpha, \beta)$.

Proof. When $j = 0$, this is [11, Theorem 12.4.2]; we thus assume hereafter that $j \in J$. The proof is similar to those of [11, Theorems 12.2.2 and 12.3.1]; for the benefit of the reader, we fill in some of the key details.

By Lemma 2.3.2, we may enlarge K as needed; in particular, we may reduce to the case $|u_j| = 1$ as in the proof of Theorem 2.2.6. Since the decomposition is unique if it exists, it is sufficient to exhibit it on an open cover of (α, β) and then glue. That is, it suffices to work in a neighborhood of any fixed $\gamma \in (\alpha, \beta)$; again, we may enlarge K to reduce to the case $\gamma = 1$.

Suppose first that $f_i^{(j)}(M, 0) > -\log \omega$. Set notation as in the proof of Theorem 2.2.6. For some sufficiently small $c > 1$, we can choose $\gamma_1 \in (\alpha, 1)$ and $\gamma_2 \in (1, \beta)$ such that the coefficient of T^{d-i} in the characteristic polynomial $Q(T)$ of N_c computes $F_i^{(j)}(M, r)$ for $r \in [-\log \gamma_2, -\log \gamma_1]$; by (a), we may apply Lemma 2.3.3 (after changing γ_1, γ_2 slightly) to deduce that this coefficient is a unit in $K\langle \gamma_1/t, t/\gamma_2 \rangle$. By (b), we can apply [11, Theorem 2.2.2] to factor $Q = Q_2 Q_1$ so that the roots of Q_1 are the i largest roots of Q under $|\cdot|_\gamma$ for all $\gamma \in [\gamma_1, \gamma_2]$. (This is true for all γ simultaneously because the construction is purely algebraic and [11, Theorem 2.2.2] takes care of convergence of the procedure.)

Use the basis B_c to identify M with $K\langle \gamma_1/t, t/\gamma_2 \rangle^d$. Then we obtain a short exact sequence

$$0 \rightarrow \text{Ker}(Q_1(N_c)) \rightarrow M \rightarrow \text{Coker}(Q_2(N_c)) \rightarrow 0$$

of free modules over $K\langle \gamma_1/t, t/\gamma_2 \rangle$. (The quotient is free because by Lemma 2.3.4 applied after rescaling, Q_1 and Q_2 generate the unit ideal in $K\langle \gamma_1/t, t/\gamma_2 \rangle[T]$.) Applying Lemma 2.2.3 to both factors (again for $c > 1$ sufficiently small, and a choice of γ_1, γ_2 depending on c), we construct a basis of M on which ∂_j acts via a matrix

$$N'_c = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}$$

for which the following conditions hold.

- (a) The matrix A_c is invertible and $|A_c^{-1}|_\gamma \cdot \max\{|\partial_j|_\gamma, |B_c|_\gamma, |C_c|_\gamma, |D_c|_\gamma\} < 1$ for all $\gamma \in [\gamma_1, \gamma_2]$.
- (b) The Newton slopes of A_c under $|\cdot|_\gamma$ account for the first i subsidiary radii of $M \otimes F_\gamma$ for all $\gamma \in [\gamma_1, \gamma_2]$.

By [11, Lemma 6.7.1], M admits a differential submodule accounting for the last $n - i$ subsidiary radii of $M \otimes F_\gamma$ for all $\gamma \in [\gamma_1, \gamma_2]$. By repeating this argument for M^\vee , we obtain the desired splitting.

To deduce the theorem in the case $p > 0$ without assuming that $f_i^{(j)}(M, 0) > \frac{1}{p-1} \log p$, we prove the theorem in the case when $f_i^{(j)}(M, 0) > \frac{1}{p^n(p-1)} \log p$ by induction on n , using ∂_j -Frobenius pushforward. This is sufficient because (b) forces $f_i^{(j)}(M, 0) > 0$, so there exists some n for which $f_i^{(j)}(M, 0) > \frac{1}{p^n(p-1)} \log p$. \square

Caution 2.3.6. In Theorem 2.3.5, M is only a locally free coherent sheaf and need not be free, because the annulus on which we are working is not closed. Even if M is free, the summands need not be free unless K is spherically complete, in which case any locally free coherent sheaf on $A_K^1(\alpha, \beta)$ is free.

Remark 2.3.7. In [11, Chapter 12], the analogous development starts with a full decomposition theorem over a closed annulus [11, Theorem 12.2.2]. We cannot do this here because we have not established an analogue of subharmonicity [11, Theorem 11.3.2(c)] for ∂_j -differential modules, except in the case $p = 0$ (see Theorems 2.7.10 and 2.7.11). We can however recover partial decomposition theorems over a closed disc or annulus, analogous to [11, Theorems 12.5.1 and 12.5.2], as follows.

Lemma 2.3.8. (a) For $x \in K[[t]]_0$ nonzero, x is a unit if and only if $|x|_{e^{-r}}$ is constant in a neighborhood of $r = 0$.

(b) For $x \in \cup_{\alpha \in (0,1)} K\langle \alpha/t, t \rangle_0$ nonzero, x is a unit if and only if the function $r \mapsto \log |x|_{e^{-r}}$ is affine in some neighborhood of 0.

Proof. We may assume that $|x|_1 = 1$. For (a), this means that $x \in \mathfrak{o}_K[[t]]$. Hence, $x = \sum_{i=0}^{\infty} a_i t^i$ is a unit if and only if a_0 is a unit in \mathfrak{o}_K , which is equivalent to $|x|_{e^{-r}}$ being constant in a neighborhood of $r = 0$. For (b), by [11, Lemma 8.2.6(c)], x is a unit if and only if its image modulo \mathfrak{m}_K in $k((t))$ is a unit or equivalently nonzero, which is equivalent to the function $r \mapsto \log |x|_{e^{-r}}$ being affine in some neighborhood of 0. \square

Theorem 2.3.9. Fix $j \in J^+$. Let M be a ∂_j -differential module of rank d over $A_K^1(\alpha, \beta)$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

(a) The function $F_i^{(j)}(M, r)$ is affine for $-\log \beta \leq r < -\log \alpha$.

(b) We have $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $-\log \beta \leq r < -\log \alpha$.

Then for any $\gamma \in (\alpha, \beta)$, $M \otimes K\langle \gamma/t, t/\beta \rangle_0$ admits a direct sum decomposition separating the first i subsidiary ∂_j -radii of $M \otimes F_\eta$ for $\eta \in [\gamma, \beta)$.

Proof. We first obtain a decomposition of $M \otimes K\langle \delta/t, t/\beta \rangle_0$ for some uncontrolled $\delta \in (\alpha, \beta)$, by arguing as in Theorem 2.3.5, but using Lemma 2.3.8(b) instead of Lemma 2.3.3. (So far we have not used condition (a).) To get the desired result, it suffices to do so for $\gamma \in (\alpha, \delta)$. For this, we use the fact that the decomposition of M over $A_K^1(\alpha, \beta)$ given by Theorem 2.3.5 is unique, so we may thus glue together the decomposition of $M \otimes K\langle \delta/t, t/\beta \rangle_0$ with the decomposition from Theorem 2.3.5. More explicitly, this involves applying Lemma 2.3.2 to the following situation: for any $\epsilon \in (\delta, \beta)$, we have

$$K\langle \gamma/t, t/\epsilon \rangle \cap K\langle \delta/t, t/\beta \rangle_0 = K\langle \gamma/t, t/\beta \rangle_0$$

within $K\langle \delta/t, t/\epsilon \rangle$. \square

Theorem 2.3.10. Fix $j \in J^+$. Let M be a ∂_j -differential module of rank d over $K\langle t/\beta \rangle$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

(a) The function $F_i^{(j)}(M, r)$ is constant in a neighborhood of $r = -\log \beta$.

(b) We have $f_i^{(j)}(M, -\log \beta) > f_{i+1}^{(j)}(M, -\log \beta)$.

Then $M \otimes K[[t/\beta]]_0$ admits a direct sum decomposition separating the first i subsidiary ∂_j -radii of $M \otimes F_\eta$ for $\eta \in (0, \beta)$.

Proof. Similar to Theorem 2.3.5, but using Lemma 2.3.8(a) instead of Lemma 2.3.3. \square

Remark 2.3.11. In Theorems 2.3.9 and 2.3.10, if K is discrete and $\beta \in |K^\times|^\mathbb{Q}$, we can begin with free differential modules over the rings $K\langle \alpha/t, t/\beta \rangle_0$ and $K[[t/\beta]]_0$, respectively. (The main reason for the restrictive hypotheses is to ensure that the resulting rings are noetherian; among other reasons, this is needed to ensure that we may freely pass between finite projective modules and finite locally free modules.) Note that this requires extending the definition of $f_i^{(j)}(M, r)$ to $r = -\log \beta$, using the completion of $\text{Frac } K[[t/\beta]]_0$ for the β -Gauss norm instead of F_β . (Compare [11, Remark 12.5.4].)

2.4 Variation for multiple derivations

In this subsection, we study the variation of intrinsic generic radii of a differential module over a disc or annulus. The results here more closely match those of [11] than in the case of a ∂_j -differential module with $j \in J$.

We first introduce a rotation construction, in the manner of [8].

Notation 2.4.1. Fix $\eta_+ \in \mathbb{R}^+$. Assume that $|u_J| = 1$. Denote \tilde{K} to be the completion of $K(x_J)$ with respect to the $(\eta_+^{-1}, \dots, \eta_+^{-1})$ -Gauss norm; view \tilde{K} as a differential field of order m with derivations $\partial_1, \dots, \partial_m$. We may use Taylor series (as in Lemma 1.2.12) to define, for any $\eta_- \in [0, \eta_+)$, an injective homomorphism $\tilde{f}^* : K\langle \eta_-/t, t/\eta_+ \rangle \rightarrow \tilde{K}\langle \eta_-/t, t/\eta_+ \rangle$ such that $\tilde{f}^*(u_j) = u_j + x_j t$.

For $\eta \in [0, \eta_+)$, we use \tilde{F}_η to denote the completion of $\tilde{K}(t)$ with respect to the η -Gauss norm. Then \tilde{f}^* extends to an injective isometric homomorphism $\tilde{f}^* : F_\eta \hookrightarrow \tilde{F}_\eta$.

Lemma 2.4.2. For any subinterval I of $[0, \eta_+)$ and any ∂_{J^+} -differential module M on $A_K^1(I)$, $\tilde{f}^* M$ gives a ∂_0 -differential module on $A_{\tilde{K}}^1(I)$. Moreover, for $\eta \in I$,

$$R_{\partial_0}(M \otimes \tilde{F}_\eta) = \min \{ \eta I R_{\partial_0}(M \otimes F_\eta); \eta_+ I R_{\partial_j}(M \otimes F_\eta) \quad (j \in J) \}.$$

Proof. This follows from the fact that

$$\partial_0|_{\tilde{f}^* M} = \partial_0|_M + \sum_{j \in J} x_j \partial_j|_M,$$

after accounting for the different normalizations. \square

Notation 2.4.3. Let M be a ∂_{J^+} -differential module of rank d on $K\langle\alpha/t, t/\beta\rangle$. For $r \in [-\log \beta, -\log \alpha]$ and $i \in \{1, \dots, d\}$, denote

$$f_i(M, r) = -\log IR(M \otimes F_{e^{-r}}; i), \quad F_i(M, r) = f_1(M, r) + \dots + f_i(M, r).$$

Note that we have changed the normalization from Notation 2.2.4, as we are now using intrinsic rather than extrinsic radii.

Theorem 2.4.4. *Let M be a ∂_{J^+} -differential module of rank d on $A_K^1[\alpha, \beta]$.*

- (a) *(Linearity) For $i = 1, \dots, d$, the functions $f_i(M, r)$ and $F_i(M, r)$ are continuous and piecewise affine.*
- (b) *(Integrality) If $i = d$ or $f_i(M, r_0) > f_{i+1}(M, r_0)$, then the slopes of $F_i(M, r)$ in some neighborhood of r_0 belong to \mathbb{Z} . Consequently, the slopes of each $f_i(M, r)$ and $F_i(M, r)$ belong to $\frac{1}{d}\mathbb{Z} \cup \dots \cup \frac{1}{d}\mathbb{Z}$.*
- (c) *(Monotonicity) Suppose that $\alpha = 0$. Then the slopes of $F_i(M, r)$ are nonpositive, and each $F_i(M, r)$ is constant for r sufficiently large.*
- (d) *(Convexity) For $i = 1, \dots, d$, the function $F_i(M, r)$ is convex.*

Proof. Before proceeding, we reduce to the case $|u_J| = 1$ as in the proof of Theorem 2.2.6. (Note that when enlarging K , we do not retain the derivations with respect to any added parameters.)

Step 1: In this step, we prove that for $i = 1, \dots, d$, $f_i(M, r)$ and $F_i(M, r)$ are continuous at $r = -\log \beta$. Moreover, if $f_i(M, -\log \beta) > 0$, we show that there exists $\gamma \in [\alpha, \beta]$ such that (a) and (b) hold for $r \in [-\log \beta, -\log \gamma]$. As in the proof of Theorem 2.2.6, we may reduce to the case $\beta = 1$.

Let R denote the completion of $\mathfrak{o}_K((t)) \otimes_{\mathfrak{o}_K} K$ for the 1-Gauss norm; note that this contains both F_1 and $K\langle\gamma/t, t\rangle_0$ for any $\gamma \in [\alpha, 1)$. We first apply Theorem 2.2.5 (if $j = 0$) or Theorem 2.2.6 (if $j \in J$), and Theorem 2.3.9, to decompose

$$M \otimes K\langle\gamma/t, t\rangle_0 = \bigoplus_{\lambda=1}^{d'} M_\lambda^{[\gamma, 1]}$$

for some $\gamma \in [\alpha, 1)$, in such a manner that the following conditions hold for $j \in J^+$ and $\lambda = 1, \dots, d'$.

- (i) The module $M_\lambda^{[\gamma, 1]} \otimes R$ is the base extension to R of a differential submodule M'_λ of $M \otimes F_1$ of pure intrinsic ∂_j -radii.
- (ii) For $\mu = 1, \dots, \text{rank } M_\lambda^{[\gamma, 1]}$ the function $f_\mu^{(j)}(M_\lambda^{[\gamma, 1]}, r)$ tends to $-\log IR_{\partial_j}(M'_\lambda)$ as $r \rightarrow 0^+$. If $j = 0$ or $IR_{\partial_j}(M'_\lambda) < 1$, then also $f_\mu^{(j)}(M_\lambda^{[\gamma, 1]}, r)$ is affine for $r \in (0, -\log \gamma]$.

This alone suffices to imply continuity of $f_i(M, r)$ and $F_i(M, r)$ at $r = 0$.

Applying Theorem 2.3.5 after possibly making γ closer to 1, we get a further decomposition $M_\lambda^{[\gamma, 1]} = \bigoplus_{\mu=1}^{d_\lambda} M_{\lambda, \mu}^{[\gamma, 1]}$ over $A_K^1[\gamma, 1)$ such that the following conditions hold for $\lambda = 1, \dots, d'$.

- (iii) For $j \in J^+$, $\mu = 1, \dots, d_\lambda$, if $IR_{\partial_j}(M'_\lambda) < 1$, then $M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_{e^{-r}}$ has pure intrinsic ∂_j -radii for $r \in (0, -\log \gamma]$.
- (iv) If $IR(M'_\lambda) < 1$, then for $j \in J^+$, $\mu = 1, \dots, d_\lambda$, ∂_j is dominant for $M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_{e^{-r}}$ for some $r \in (0, -\log \gamma]$ if and only if the same holds for all $r \in (0, -\log \gamma]$.
- (v) If $\lambda, \lambda' \in \{1, \dots, d'\}$ satisfy $IR(M'_\lambda) > IR(M'_{\lambda'})$, then $IR(M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_{e^{-r}}) > IR(M_{\lambda', \mu'}^{[\gamma, 1]} \otimes F_{e^{-r}})$ for all $\mu \in \{1, \dots, d_\lambda\}$, $\mu' \in \{1, \dots, d_{\lambda'}\}$ and $r \in (0, -\log \gamma]$.

The piecewise affinity from (a) in the case $f_i(M, 0) > 0$ now follows from Theorems 2.2.5(a) and 2.2.6(a) applied to each $M_{\lambda, \mu}^{[\gamma, 1]}$.

To check (b), it suffices to verify integrality of slope times rank for each component $M_{\lambda, \mu}^{[\gamma, 1]}$ for which $IR(M'_\lambda) < 1$. If ∂_0 is dominant for $M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_{e^{-r}}$ for some (hence all) $r \in (0, -\log \gamma]$, (b) follows from Theorem 2.2.5(b). Otherwise, pick arbitrary $\eta_- < \eta_+ \in (\gamma, 1)$ such that for $\eta \in (\eta_-, \eta_+)$,

$$\eta_-/\eta_+ > IR(M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_\eta) / IR_{\partial_0}(M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_\eta).$$

Define \tilde{K} as in Notation 2.4.1. By Lemma 2.4.2, for $\eta \in (\eta_-, \eta_+)$, we have

$$\begin{aligned} R_{\partial_0}(\tilde{f}^* M_{\lambda, \mu}^{[\gamma, 1]} \otimes \tilde{F}_\eta) &= \min \left\{ \eta IR_{\partial_0}(M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_\eta); \eta_+ IR_{\partial_j}(M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_\eta) \quad (j \in J) \right\} \\ &= \eta_+ IR(M_{\lambda, \mu}^{[\gamma, 1]} \otimes F_\eta). \end{aligned}$$

In particular, $(f_1^{(0)})'(\tilde{f}^* M_{\lambda, \mu}^{[\gamma, 1]} \otimes \tilde{F}_\eta, -\log \eta) = f_1'(M_{\lambda, \mu}^{[\gamma, 1]}, -\log \eta) = (f_1)'_-(M_{\lambda, \mu}^{[\gamma, 1]}, 0)$ for $\eta \in (\eta_-, \eta_+)$. (Note that we showed in the proof of (a) that $f_1(M_{\lambda, \mu}^{[\gamma, 1]}, r)$ extends continuously to $r = 0$, so its left derivative at 0 makes sense.) Thus, the statement (b) follows by applying Theorem 2.2.5(b) to $\tilde{f}^* M_{\lambda, \mu}^{[\gamma, 1]}$.

Step 1': As a corollary of step 1, we deduce that for any $r_0 \in [-\log \beta, -\log \alpha]$, $f_i(M, r)$ and $F_i(M, r)$ are continuous at r_0 , and in case $f_i(M, r_0) > 0$ one also has (a) and (b) in a neighborhood of r_0 . (In particular, we will then have continuity of $f_i(M, r)$ and $F_i(M, r)$ over all of $[-\log \beta, -\log \alpha]$.) To make this deduction, we first replace β by $\gamma = e^{-r_0}$ in case $r_0 < -\log \alpha$, to obtain all the desired assertions in a right neighborhood of r_0 . By pulling back along $t \mapsto t^{-1}$ and then repeating the argument, we obtain the desired assertions in a left neighborhood of r_0 .

Step 2: In this step, we prove that (d) holds in a neighborhood of each $r_0 \in (-\log \beta, -\log \alpha)$ for which $f_i(M, r_0) > 0$. It suffices to check in the case $f_i(M, r_0) > f_{i+1}(M, r_0)$, as the general case follows by interpolation.

At this point, we may reduce to the case $r_0 = 0$. As in Step 1, for some $\eta_- \in (\alpha, \eta)$, we have a partial decomposition of M over $K\langle \eta_-/t, t \rangle_0$ as $M = \bigoplus_{\lambda_- = 1}^{d_-} M_{\lambda_-}^{[\eta_-, 1]}$ satisfying (i) and (ii). For some $\eta_+ \in (1, \beta)$, we also have a partial decomposition over $K\langle \eta_+^{-1}/t, t \rangle_0$ of the pullback of M along $t \mapsto t^{-1}$ as $M = \bigoplus_{\lambda_+ = 1}^{d_+} M_{\lambda_+}^{[1, \eta_+]}$ satisfying appropriate analogues of (i) and (ii). By making η_- and η_+ closer to 1, we may guarantee that for each index λ_- (resp. λ_+) for which the ratio $IR(M'_{\lambda_-})/IR_{\partial_0}(M'_{\lambda_-})$ (resp. $IR(M'_{\lambda_+})/IR_{\partial_0}(M'_{\lambda_+})$) is less than 1, this ratio is also less than η_-/η_+ .

Use Notation 2.4.1; by Theorem 2.2.5, $F_i^{(0)}(\tilde{f}^*M, r)$ is convex at $r = 0$. In particular, $(F_i^{(0)})'_-(\tilde{f}^*M, 0) \leq (F_i^{(0)})'_+(\tilde{f}^*M, 0)$. It suffices to show that

$$(F_i^{(0)})'_+(\tilde{f}^*M, 0) - \theta_i(M, 0) \leq (F_i)'_+(M, 0) \quad (2.4.4.1)$$

$$(F_i^{(0)})'_-(\tilde{f}^*M, 0) - \theta_i(M, 0) \geq (F_i)'_-(M, 0), \quad (2.4.4.2)$$

where $\theta_i(M, 0)$ denotes the sum of the dimensions of the constituents N of $M \otimes F_1$ for which ∂_0 is dominant and $f_1(N, 0) \geq f_i(M, 0)$.

The proofs of (2.4.4.1) and (2.4.4.2) are similar, so we focus on (2.4.4.1). Decompose M as in Step 1. For each λ such that ∂_0 is dominant for M'_λ , we have by Lemma 2.4.2 that in a punctured right neighborhood of $r = 0$,

$$F_1^{(0)}(\tilde{f}^*M_{\lambda, \mu}^{[\gamma, 1]}, r) = F_1^{(0)}(M_{\lambda, \mu}^{[\gamma, 1]}, r)$$

and so

$$(F_1^{(0)})'_+(\tilde{f}^*M_{\lambda, \mu}^{[\gamma, 1]}, 0) - 1 = (F_1^{(0)})'_+(M_{\lambda, \mu}^{[\gamma, 1]}, 0) - 1 \leq (F_1)'_+(M_{\lambda, \mu}^{[\gamma, 1]}, 0).$$

(The term -1 comes from the change of normalization from Notation 2.2.4 to Notation 2.4.3. The inequality can be strict if ∂_j is also dominant for M'_λ for some $j > 0$.) For each λ such that ∂_0 is not dominant for M'_λ , we have by Lemma 2.4.2 (and the choice of η_+, η_-) that in a punctured right neighborhood of $r = 0$,

$$F_1^{(0)}(\tilde{f}^*M_{\lambda, \mu}^{[\gamma, 1]}, r) = F_1^{(j)}(M_{\lambda, \mu}^{[\gamma, 1]}, r) - \log \eta_+$$

and so

$$(F_1^{(0)})'_+(\tilde{f}^*M_{\lambda, \mu}^{[\gamma, 1]}, 0) = (F_1)'_+(M_{\lambda, \mu}^{[\gamma, 1]}, 0).$$

Summing over components yields (2.4.4.1).

Step 3: In this step, we prove (a), (b), (d) in general, by induction on i . Keep in mind that we already have the continuity aspect of (a) in general (by Step 1'), and all of (a), (b), (d) in a neighborhood of any $r_0 \in [-\log \beta, -\log \alpha]$ for which $f_i(M, r_0) > 0$ (by Steps 1, 1', 2).

We first check the piecewise affinity aspect of (a) in a right neighborhood of some r_0 for which $f_i(M, r_0) = 0$. By the induction hypothesis, we can pick $r_1 > r_0$ such that $F_{i-1}(M, r)$ is affine on $[r_0, r_1]$. Suppose that $r_2 \in (r_0, r_1)$ is a value for which $f_i(M, r_2) > 0$. By continuity of f_i , there exists an open neighborhood of r_2 on which $f_i(M, r)$ is everywhere positive. Let U be the union of all such neighborhoods in $[r_0, r_1]$; then U is an open interval (r_3, r_4) , and

$f_i(M, r_3) = 0$. Since (a) and (d) hold in a neighborhood of each $r \in U$, $F_i(M, r)$ and hence $f_i(M, r)$ are piecewise affine and convex on U . In order for $f_i(M, r)$ to both be convex and to tend to 0 as $r \rightarrow r_3^+$, $f_i(M, r)$ must have no nonpositive slopes; that is, $f_i(M, r)$ is strictly increasing on U . However, we must also have $f_i(M, r_4) = 0$ unless $r_4 = r_1$. The former possibility leads to a contradiction, so we must have $r_4 = r_1$.

To sum up the previous paragraph, we now know that if there exists $r_2 \in (r_0, r_1]$ such that $f_i(M, r_2) > 0$, then $f_i(M, r) > 0$ for all $r \in [r_2, r_1]$. Consequently, on some right neighborhood of r_0 , $f_i(M, r)$ is either everywhere zero or everywhere positive. In the former case, $f_i(M, r)$ is clearly affine on a right neighborhood of r_0 . In the latter case, pick $r_2 \in (r_0, r_1]$ for which $f_i(M, r_2) > 0$; then the slopes of $f_i(M, r)$ on $(r_0, r_2]$ are nondecreasing, bounded below by 0, and (by (b)) confined to a discrete subset of \mathbb{R} . Consequently, there must be a least slope achieved, occurring on a right neighborhood of r_0 . We thus deduce (a) in a right neighborhood of r_0 . By symmetry, the same argument applies to left neighborhoods; we may thus deduce (a) in general.

Since (a) is now known, $f_i(M, r)$ takes only finitely many slopes on all of $[-\log \beta, -\log \alpha]$. Except possibly for the slope 0, each slope must occur at some r for which $f_i(M, r) > 0$; consequently, the knowledge of (b) at such points now implies (b) in general.

Finally, we still need to check (d) in a neighborhood of a point r_0 at which $f_i(M, r_0) = 0$. By (a), $f_i(M, r)$ is affine on a right neighborhood of r_0 and on a left neighborhood of r_0 ; since $f_i(M, r) \geq 0$ everywhere, the right slope of $f_i(M, r)$ at r_0 must be greater than or equal to the left slope of $f_i(M, r)$ at r_0 . Since the same is true of $F_{i-1}(M, r)$ by the induction hypothesis, the same must also be true of $F_i(M, r)$. This yields (d).

Step 4: In this step, we prove (c). By Dwork's transfer theorem (see Proposition 1.2.14), for any $\eta < R_{\partial_0}(M \otimes F_\beta)$, $M \otimes K\langle t/\eta \rangle$ admits a basis in the kernel of ∂_0 . In other words, $M \otimes K\langle t/\eta \rangle$ is isomorphic to the pullback of a (∂_J) -differential module over K . Consequently, $F_i(M, r)$ is constant for r sufficiently large; by (d), this implies that $F_i(M, r)$ has all slopes nonpositive. \square

Remark 2.4.5. If $p = 0$, then the assertion that $r^{\dim V_r} \in |K^\times|$ in Theorem 1.5.6 implies that $d!F_i(M, r) \in \log |K^\times| + \mathbb{Z}r$. If $p > 0$, then we only deduce that for h a nonnegative integer,

$$f_i(M, r) > \frac{p^{-h}}{p-1} \log p \implies d!F_i(M, r) \in p^{-h} \log |K^\times| + \mathbb{Z}r.$$

In either case, we may conclude that the values of r at which $F_i(M, r)$ changes slope must belong to $\mathbb{Q} \log |K^\times|$.

2.5 Decomposition for multiple variations

We now obtain decomposition theorems which allow for multiple derivations.

Theorem 2.5.1. *Let M be a ∂_{J^+} -differential module of rank d on $A_K^1(\alpha, \beta)$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

- (a) *The function $F_i(M, r)$ is affine for $-\log \beta < r < -\log \alpha$.*

(b) We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log \beta < r < -\log \alpha$.

Then M admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (\alpha, \beta)$.

Proof. Before proceeding, we reduce to the case $|u_J| = 1$ as in the proof of Theorem 2.3.5. It suffices to prove the decomposition in a neighborhood of each $r_0 \in (-\log \beta, -\log \alpha)$. Again, we may assume $r_0 = 0$.

We continue with Step 2 in the proof of Theorem 2.4.4. We may further impose the auxiliary condition that

$$-\log(\eta_-) < f_i(M, 0) - f_{i+1}(M, 0). \quad (2.5.1.1)$$

By (2.4.4.1) and the symmetric result, we have

$$(F_i)'_-(M, 0) \leq (F_i^{(0)})'_-(\tilde{f}^*M, 0) - \theta_i(M, 0) \leq (F_i^{(0)})'_+(\tilde{f}^*M, 0) - \theta_i(M, 0) \leq (F_i)'_+(M, 0); \quad (2.5.1.2)$$

all the inequalities are forced to be equalities as $F_i(M, r)$ is affine in a neighborhood of $r = 0$. In particular, $F_i^{(0)}(\tilde{f}^*M, r)$ is affine when $r \in (-\log \eta_+, -\log \eta_-]$. We would get the decomposition by Theorem 2.3.5 if we knew that $f_i^{(0)}(\tilde{f}^*M, r) > f_{i+1}^{(0)}(\tilde{f}^*M, r)$ for r in a neighborhood of $r = 0$. Indeed, by our auxiliary condition (2.5.1.1) and Lemma 2.4.2,

$$f_i^{(0)}(\tilde{f}^*M, 0) > \log(\eta_-) + f_i(M, 0) > f_{i+1}(M, 0) \geq f_{i+1}^{(0)}(\tilde{f}^*M, 0).$$

The theorem follows. \square

Theorem 2.5.2. Let M be a ∂_{J^+} -differential module of rank d on $A_K^1[0, \beta)$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

- (a) The function $F_i(M, r)$ is affine for $r > -\log \beta$. (This implies $F_i(M, r)$ is constant by Theorem 2.4.4(c).)
- (b) We have $f_i(M, r) > f_{i+1}(M, r)$ for all (some) $r > -\log \beta$.

Then M admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (0, \beta)$.

Proof. Before proceeding, we reduce to the case $|u_J| = 1$ as in the proof of Theorem 2.3.5. As noted in Step 4 of the proof of Theorem 2.4.4, there exists some $\eta \in (0, \beta)$ such that $M \otimes K\langle t/\eta \rangle$ is isomorphic to the pullback of a (∂_J) -differential module M_0 over K . Consequently, we have the desired decomposition of M over $A_K^1[0, \eta]$ by pulling back the decomposition of M_0 in the sense of Theorem 1.4.21. The theorem follows by applying Theorem 2.5.1 to $A_K^1(\eta', \beta)$ for some $\eta' \in (0, \eta)$. \square

Remark 2.5.3. We can sometimes verify the hypotheses of Theorem 2.5.2 using monotonicity and convexity (Theorem 2.4.4(c) and (d)). For example, if $F_i'(M, r_0) = 0$, then $F_i(M, r)$ is constant for $r \geq r_0$. Moreover, if we also have $f_i(M, r_0) > f_{i+1}(M, r_0)$, then condition (b) holds for $r \geq r_0$.

Remark 2.5.4. As in Remark 2.3.7, we cannot state a decomposition theorem over a closed annulus without assuming $p = 0$ (in which case see Theorems 2.7.12 and 2.7.13). However, we do get partial decomposition theorems analogous to Theorems 2.7.10 and 2.7.11, as follows.

Theorem 2.5.5. *Let M be a ∂_{J^+} -differential module of rank d on $A_K^1(\alpha, \beta]$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

- (a) *The function $F_i(M, r)$ is affine for $-\log \beta \leq r < -\log \alpha$.*
- (b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log \beta \leq r < -\log \alpha$.*

Then for any $\gamma \in (\alpha, \beta)$, $M \otimes K\langle \gamma/t, t/\beta \rangle_0$ admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (\gamma, \beta)$.

Proof. The fact that this holds for a single γ , even without hypothesis (a), is a corollary of Step 1 of the proof of Theorem 2.4.4. The desired conclusion follows by combining this assertion with Theorem 2.5.1. \square

Theorem 2.5.6. *Let M be a ∂_{J^+} -differential module of rank d on $A_K^1[0, \beta]$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

- (a) *The function $F_i(M, r)$ is affine for $r \geq -\log \beta$.*
- (b) *We have $f_i(M, -\log \beta) > f_{i+1}(M, -\log \beta)$.*

Then $M \otimes K[[t/\beta]]_0$ admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (0, \beta)$.

Proof. This follows by combining Theorems 2.5.2 and 2.5.5. \square

Remark 2.5.7. As in Remark 2.3.11, if K is discretely valued and $\beta \in |K^\times|^\mathbb{Q}$, we can admit modules in Theorems 2.5.5 and 2.5.6 defined directly over the corresponding rings of bounded functions, namely $K\langle \alpha/t, t/\beta \rangle_0$ and $K[[t/\beta]]_0$.

2.6 An application to Swan conductors

As promised earlier (Remark 1.5.10), we can use the results of this section to extend the results of [8] by relaxing [8, Hypothesis 2.1.3] to the hypothesis that K is of rational type. As this is straightforward to do, we merely summarize the outcome by stating and deducing a result which includes [8, Theorems 2.7.2 and 2.8.2].

Theorem 2.6.1. *Let M be a differential module of rank d on $A_K^1(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$, such that $IR(M \otimes F_\rho) \rightarrow 1$ as $\rho \rightarrow 1^-$. (That is, M is solvable at 1.) Then for some $\eta \in (0, 1)$, there exist a decomposition $M = M_1 \oplus \dots \oplus M_r$ on $A_K^1(\eta, 1)$ and nonnegative rational numbers b_1, \dots, b_r with $\sum_i b_i \cdot \text{rank}(M_i) \in \mathbb{Z}$, such that*

$$IR(M_i \otimes F_\rho; j) = \rho^{b_i} \quad (i = 1, \dots, r; \quad j = 1, \dots, \text{rank}(M_i)).$$

Proof. By Theorem 2.4.4, for $l = 1, \dots, d$, the function $d!F_i(M, r)$ on $(0, -\log \eta)$ is continuous, convex, and piecewise affine with integer slopes. By hypothesis, $d!F_i(M, r) \rightarrow 0$ as $r \rightarrow 0^+$; because of this and the fact that $d!F_i(M, r) \geq 0$ for all r , the slopes of $F_i(M, r)$ are forced to be nonnegative. Hence there is a least such slope, that is, $d!F_i(M, r)$ is linear in a right neighborhood of $r = 0$.

We can thus choose η so that $d!F_i(M, r)$ is linear on $(0, -\log \eta)$ for $i = 1, \dots, d$. We obtain the desired decomposition by Theorem 2.5.2; the integrality of $\sum_i b_i \cdot \text{rank}(M_i)$ follows from the fact that $F_d(M, r)$ has integral slopes, again by Theorem 2.4.4. \square

2.7 Subharmonicity for residual characteristic 0

When $m = 0$, the functions $F_i(M, r)$ obey a certain subharmonicity property [11, Theorem 11.3.2]. When the residual characteristic p is equal to 0, one can obtain a similar result even when K carries derivations. (See Remark 2.2.8 for discussion of the case $p > 0$.)

Hypothesis 2.7.1. Throughout this subsection, we assume $p = 0$.

Definition 2.7.2. For $\bar{\mu} \in (k^{\text{alg}})^\times$, let μ be a lift of $\bar{\mu}$ in some finite extension L of K . Let E be a finite extension of the completion of $\mathfrak{o}_K[t]_{(t)} \otimes_{\mathfrak{o}_K} L$ for the 1-Gauss norm. For $\alpha \leq 1 \leq \beta$, define the substitution

$$T_\mu : K\langle \alpha/t, t/\beta \rangle \rightarrow E, \quad t \mapsto t + \mu.$$

Definition 2.7.3. Fix $j \in J^+$. Let M be a ∂_j -differential module of rank d on $A_K^1[\alpha, \beta]$ for some $\alpha \leq 1 \leq \beta$. For $i = 1, \dots, n$, let $s_{\infty, i}^{(j)}(M)$ and $s_{0, i}^{(j)}(M)$ be the left (if $\beta \neq 1$) and right (if $\alpha \neq 1$) slopes of $F_i^{(j)}(M, r)$ at $r = 0$. For $\bar{\mu} \in (k^{\text{alg}})^\times$, pick any $\mu \in \mathfrak{o}_L$ lifting $\bar{\mu}$ in a finite unramified extension L of K , and let $s_{\bar{\mu}, i}^{(j)}(M)$ be the right slope of $F_i^{(j)}(T_\mu^*(M), r)$ at $r = 0$. Note that $T_\mu^*(M)$ is still a ∂_j -differential module by Lemma 1.4.5.

If M is a ∂_{J^+} -differential module of rank d on $A_K^1[\alpha, \beta]$ for some $\alpha \leq 1 \leq \beta$, for $i = 1, \dots, n$ and $\bar{\mu} \in k^{\text{alg}}$, we similarly define $s_{\infty, i}(M)$ and $s_{\bar{\mu}, i}(M)$ as the slopes of the corresponding functions $F_i(M, r)$ or $F_i(T_\mu^*(M), r)$.

Theorem 2.7.4. Fix $j \in J^+$. Let M be a ∂_j -differential module of rank d on $A_K^1[\alpha, \beta]$ for some $\alpha < 1 < \beta$. Choose $i \in \{1, \dots, d\}$ such that $f_i^{(j)}(M, 0) > 0$.

- (a) The quantity $s_{\bar{\mu}, i}^{(j)}(M)$ does not depend on the lift μ and the unramified extension L/K .
- (b) We have $s_{\bar{\mu}, i}^{(j)}(M) \leq 0$ for all $\bar{\mu} \neq 0$, with equality for all but finitely many $\bar{\mu}$.
- (c) We have

$$s_{\infty, i}^{(j)}(M) \leq \sum_{\bar{\mu} \in k^{\text{alg}}} s_{\bar{\mu}, i}^{(j)}(M),$$

with equality if either $i = n$ and $f_n^{(j)}(M, 0) > 0$ or $i < n$ and $f_i^{(j)}(M, 0) \geq f_{i+1}^{(j)}(M, 0)$.

Proof. When $j = 0$, this is [11, Theorem 11.3.2(d)]. When $j \in J$, the proof of Theorem 2.2.6 reduces the problem to [11, Theorem 11.2.1(c)]. Note that we do not have to use the Frobenius pushforward. \square

Remark 2.7.5. Let L be a complete extension of K such that ∂_j extends to L . Then $M \otimes L$ becomes a ∂_j -differential module over $A_L^1[\alpha, \beta]$. For $\bar{\mu} \notin k^{\text{alg}}$, we always have $s_{\bar{\mu}, i}^{(j)}(M) = 0$; this can be seen either by inspecting the proof of Theorem 2.7.4, or by deducing the claim directly from (b). Namely, (b) implies that the equality $s_{\bar{\mu}, i}^{(j)}(M) = 0$ holds with only finitely many exceptions; on the other hand, if $\bar{\mu}$ were an exception not in k^{alg} , then so would be each of its infinitely many conjugates in an algebraic closure of the residue field of L .

Theorem 2.7.6. Let M be a ∂_{J^+} -differential module of rank d on $A_K^1[\alpha, \beta]$ for some $\alpha < 1 < \beta$. Choose $i \in \{1, \dots, d\}$ such that $f_i(M, 0) > 0$.

- (a) The quantity $s_{\bar{\mu}, i}(M)$ does not depend on the lift μ and the unramified extension L/K .
- (b) We have $s_{\bar{\mu}, i}(M) \leq 0$ for all $\bar{\mu} \neq 0$, with equality for all but finitely many $\bar{\mu}$.
- (c) We have

$$s_{\infty, i}(M) \leq \sum_{\bar{\mu} \in k^{\text{alg}}} s_{\bar{\mu}, i}(M).$$

Proof. Suppose first that ∂_0 is dominant for each irreducible component of $M \otimes F_1$ which contributes to $F_i(M, 0)$. Then $s_{\infty, i}(M)$ is less than or equal to the left slope of $F_i^{(0)}(M, r)$ at $r = 0$, whereas $s_{\bar{\mu}, i}(M)$ is greater than or equal to the right slope of $F_i^{(0)}(T_\mu^*(M), r)$ at $r = 0$. We may thus reduce to the case $m = 0$, which is [11, Theorem 11.3.2(c)].

It suffices to reduce to the case where ∂_0 is dominant for each irreducible component of $M \otimes F_1$ which contributes to $F_i(M, 0)$. This proceeds as in Step 2 of the proof of Theorem 2.4.4, except that we may end up working over an enlargement of K . This causes no harm in (a) or (b), but in (c) the sum may end up running over a larger field. However, the argument of Remark 2.7.5 shows that the extra terms do not contribute: that is, we may use (b) to show that $s_{\bar{\mu}, i}(M) = 0$ if $\bar{\mu} \notin k^{\text{alg}}$, so (c) holds as written. \square

Remark 2.7.7. The proof given above does not achieve the equality in (c) for $m > 0$, because the reduction in the last paragraph does not maintain equality.

As in [11, Subsection 12.2], we can study decomposition theorems over closed annuli or discs using subharmonicity.

Definition 2.7.8. Fix $j \in J^+$. Let M be a ∂_j -differential module over $K\langle\alpha/t, t/\beta\rangle$ with $\alpha \leq 1 \leq \beta$. Define the i -th ∂_j -discrepancy of M at $r = 0$ as

$$\text{disc}_i^{(j)}(M, 0) = - \sum_{\bar{\mu} \in (k^{\text{alg}})^\times} s_{\bar{\mu}, i}^{(j)}(M);$$

it is nonnegative by Theorem 2.7.4. By Remark 2.7.5, this definition is invariant under enlarging K . We may extend the definition to general $r \in [-\log \beta, -\log \alpha]$ by pulling back M along

$$K\langle \alpha/t, t/\beta \rangle \rightarrow K(c)^\wedge \langle \alpha e^r/t, t/\beta e^r \rangle, \quad t \mapsto ct,$$

where c is transcendental over K and $K(c)^\wedge$ is the completion with respect to the e^{-r} -Gauss norm.

If M is a finite ∂_{J^+} -differential module over $K\langle \alpha/t, t/\beta \rangle$ with $\alpha \leq 1 \leq \beta$, we similarly define the i -th discrepancy $\text{disc}_i(M, 0)$ of M at $r = 0$ as the sum of $-s_{\bar{\mu}, i}(M)$ over $\bar{\mu} \in (k^{\text{alg}})^\times$. This quantity is again nonnegative, and is again invariant under enlarging K (this time by the final remark in the proof of Theorem 2.7.6). This definition can similarly be extended to $r \in [-\log \beta, -\log \alpha]$.

Remark 2.7.9. If $r \notin \mathbb{Q} \log |K^\times|$, then Remark 2.4.5 implies that $F_i(M, r)$ is affine in a neighborhood of r . By Theorem 2.7.6, it follows that $\text{disc}_i(M, r) = 0$.

Theorem 2.7.10. Fix $j \in J^+$. Let M be a ∂_j -differential module over $K\langle \alpha/t, t/\beta \rangle$ of rank d . Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

- (a) We have $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $r \in [-\log \beta, -\log \alpha]$.
- (b) The function $F_i^{(j)}(M, r)$ is affine for $r \in [-\log \beta, -\log \alpha]$.
- (c) We have $\text{disc}_i^{(j)}(M, -\log \alpha) = \text{disc}_i^{(j)}(M, -\log \beta) = 0$.

Then there is a direct sum decomposition of M inducing, for each $\eta \in [\alpha, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary ∂_j -radii from the others.

Proof. Similar to Theorem 2.3.5 but invoking [11, Lemma 12.1.3] instead. \square

Theorem 2.7.11. Fix $j \in J^+$. Let M be a ∂_j -differential module over $K\langle t/\beta \rangle$ of rank d . Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

- (a) We have $f_i^{(j)}(M, -\log \beta) > f_{i+1}^{(j)}(M, -\log \beta)$.
- (b) The function $F_i^{(j)}(M, r)$ is constant for r in a neighborhood of $-\log \beta$.
- (c) We have $\text{disc}_i^{(j)}(M, -\log \beta) = 0$.

Then there is a direct sum decomposition of M inducing, for each $\eta \in (0, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary ∂_j -radii from the others.

Proof. One can prove this similarly to Theorem 2.3.5 by invoking [11, Lemma 12.1.2] instead. It is also an immediate corollary of Theorems 2.7.10 and 2.3.10; note that Theorem 2.7.4 verifies the condition (c) in Theorem 2.7.10. \square

Theorem 2.7.12. Let M be a ∂_{J^+} -differential module over $K\langle \alpha/t, t/\beta \rangle$ of rank d . Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

(a) We have $f_i(M, r) > f_{i+1}(M, r)$ for $r \in [-\log \beta, -\log \alpha]$.

(b) The function $F_i(M, r)$ is affine for $r \in [-\log \beta, -\log \alpha]$.

(c) We have $\text{disc}_i(M, -\log \alpha) = \text{disc}_i(M, -\log \beta) = 0$.

Then there is a direct sum decomposition of M inducing, for each $\eta \in [\alpha, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary radii from the others.

Proof. Similar to Theorem 2.5.1 but invoking Theorem 2.7.10 instead on the boundary. \square

Theorem 2.7.13. Let M be a ∂_{J^+} -differential module over $K\langle t/\beta \rangle$ of rank d . Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.

(a) We have $f_i(M, -\log \beta) > f_{i+1}(M, -\log \beta)$.

(b) The function $F_i(M, r)$ is constant for r in a neighborhood of $-\log \beta$.

(c) We have $\text{disc}_i(M, -\log \beta) = 0$.

Then there is a direct sum decomposition of M inducing, for each $\eta \in (0, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary radii from the others.

Proof. It follows from Theorems 2.7.12 and 2.3.10; note also that Theorem 2.7.6 verifies the condition (c) in Theorem 2.7.12. \square

3 Differential modules on higher-dimensional spaces

We now study the variation of subsidiary radii of differential modules on some simple higher-dimensional spaces. Rather than derive these directly, we deduce these from the corresponding results on one-dimensional spaces from the previous section, using some properties of convex functions.

Throughout this section, we retain Hypothesis 2.0.1.

3.1 Convex functions

In this subsection, we set some terminology for convex functions, as in [10, Section 2].

Definition 3.1.1. For a subset $C \subseteq \mathbb{R}^n$, we denote its interior by $\text{int}(C)$. We say it is *convex* if for all $x, y \in C$ and all $t \in [0, 1]$, $tx + (1-t)y \in C$. For $C \subseteq \mathbb{R}^n$ convex, a function $f : C \rightarrow \mathbb{R}$ is *convex* if for all $x, y \in C$ and all $t \in [0, 1]$,

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y). \quad (3.1.1.1)$$

Such a function is continuous on $\text{int}(C)$.

Definition 3.1.2. An *affine functional* on \mathbb{R}^n is a map $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form $\lambda(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n + b$ for some $a_1, \dots, a_n, b \in \mathbb{R}$. If $a_1, \dots, a_n \in \mathbb{Z}$, we say λ is *transintegral* (short for “integral after translation”); if also $b \in \mathbb{Z}$, we say λ is *integral*. For $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ an affine functional, define the *slope* of λ as the linear functional $\tilde{\lambda}(x) = \lambda(x) - \lambda(0)$.

Definition 3.1.3. For $f : C \rightarrow \mathbb{R}^n$ convex, a *domain of affinity* of f is a subset U of C with nonempty interior (in \mathbb{R}^n) on which f agrees with an affine functional λ . The nonempty interior condition ensures that λ is uniquely determined; we call it the *ambient functional* on U .

Lemma 3.1.4. Let $f : C \rightarrow \mathbb{R}^n$ be a convex function, and let $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ be an affine functional which agrees with f on a subset of C with nonempty interior in \mathbb{R}^n .

- (a) We have $f(x) \geq \lambda(x)$ for all $x \in C$.
- (b) The set of $x \in C$ for which $f(x) = \lambda(x)$ is a convex subset of C .
- (c) If λ' is another affine functional with the same slope as λ , and λ' occurs as the ambient functional of some domain of affinity of f , then $\lambda = \lambda'$.

Proof. For (a), choose y in the interior of a domain of affinity U of f with ambient functional λ . For $\epsilon > 0$ sufficiently small, the quantity z defined by $\epsilon x + (1 - \epsilon)z = y$ will also belong to U . By convexity of f , $\epsilon f(x) + (1 - \epsilon)\lambda(z) \geq \lambda(y)$, so

$$f(x) \geq \frac{\lambda(y) - (1 - \epsilon)\lambda(z)}{\epsilon} = \lambda(x).$$

We may deduce (b) and (c) immediately from (a). □

Definition 3.1.5. A subset $C \subseteq \mathbb{R}^n$ is *polyhedral* if there exist finitely many affine functionals $\lambda_1, \dots, \lambda_r$ such that

$$C = \{x \in \mathbb{R}^n : \lambda_i(x) \geq 0 \quad (i = 1, \dots, r)\}. \quad (3.1.5.1)$$

(We do not require C to be bounded.) If the λ_i can all be taken to be (trans)integral, we say that C is *(trans)rational polyhedral*. (We use *RP* and *TRP* as shorthand for *rational polyhedral* and *transrational polyhedral*.) For $C \subseteq \mathbb{R}^n$ a convex subset of \mathbb{R}^n , a continuous convex function $f : C \rightarrow \mathbb{R}^n$ is *polyhedral* if there exist finitely many affine functionals $\lambda'_1, \dots, \lambda'_s$ such that

$$f(x) = \max\{\lambda'_1(x), \dots, \lambda'_s(x)\} \quad (x \in C). \quad (3.1.5.2)$$

(In particular, such a function extends continuously to a convex function on the closure of C , or even to all of \mathbb{R}^n .) Similarly, if C is (trans)rational polyhedral, we say f is *(trans)integral polyhedral* if (3.1.5.2) holds for some (trans)integral affine functionals $\lambda'_1, \dots, \lambda'_s$.

Remark 3.1.6. If C is a convex subset of \mathbb{R}^n , then a continuous convex function $f : C \rightarrow \mathbb{R}^n$ is polyhedral if and only if C is covered by finitely many domains of affinity for f , by [10, Lemma 2.2.6]. Moreover, if C is compact, then it suffices to check that every point in C has a neighborhood covered by finitely many domains of affinity for f , as then compactness will imply the existence of finitely many domains of affinity which cover C .

3.2 Detecting polyhedral functions

In this subsection, we establish a theorem that can be used to detect polyhedrality of certain convex functions based on integrality properties of certain values of the functions. We start with a weaker result in the same spirit, from [10, Section 2].

Notation 3.2.1. In this subsection, for a point $x \in \mathbb{Q}^n$, we write x_1, \dots, x_n for the coordinates of x .

Theorem 3.2.2. *Let C be a bounded RP subset of \mathbb{R}^n , and let $f : C \rightarrow \mathbb{R}$ be a continuous convex function. Then f is integral polyhedral if and only if*

$$f(x) \in \mathbb{Z} + \mathbb{Z}x_1 + \dots + \mathbb{Z}x_n \quad (x \in C \cap \mathbb{Q}^n). \quad (3.2.2.1)$$

Proof. See [10, Theorem 2.4.2]. □

One cannot hope to similarly detect transintegral polyhedral functions by sampling them at individual points, i.e., on zero-dimensional TRP subsets of \mathbb{R}^n . The best one can do is detect them by sampling on one-dimensional TRP subsets of \mathbb{R}^n , as follows.

Definition 3.2.3. Let C be a convex subset of \mathbb{R}^n . We say a function $f : C \rightarrow \mathbb{R}$ is *convex transintegral polyhedral in dimension 1* if its restriction to the intersection of C with any one-dimensional TRP subset of \mathbb{R}^n is continuous, convex, and transintegral polyhedral. In other words, for any $x \in C, a \in \mathbb{Q}^n$, if we put $I_{x,a} = \{t \in \mathbb{R} : x + ta \in C\}$, then the function $g : I_{x,a} \rightarrow \mathbb{R}$ defined by $g(t) = f(x + ta)$ is continuous, convex, piecewise affine with slopes in $a_1\mathbb{Z} + \dots + a_n\mathbb{Z}$, and has only finitely many slopes. (The latter is automatic if $I_{x,a}$ is closed and bounded, which always occurs if C is compact.)

Theorem 3.2.4. *Let C be a TRP subset of \mathbb{R}^n . Let $f : C \rightarrow \mathbb{R}$ be a function which is convex transintegral polyhedral in dimension 1. Then f itself is convex and transintegral polyhedral (hence continuous).*

The proof is somewhat complicated, and will occupy the rest of this section. We first tackle the case where C is compact, for which we assemble several lemmas.

Definition 3.2.5. Let C be a TRP subset of \mathbb{R}^n . For $x \in C$, define the *angle* of C at x , denoted $\angle_x C$, to be the set of $z \in \mathbb{R}^n$ such that for some $t_0 > 0$, $x + tz \in C$ for $t \in [0, t_0]$. It is clear that $\angle_x C$ is an RP subset of \mathbb{R}^n stable under multiplication by $\mathbb{R}_{>0}$.

Lemma 3.2.6. *Let C be a TRP subset of \mathbb{R}^n , and let $f : C \rightarrow \mathbb{R}$ be a function which is convex transintegral polyhedral in dimension 1. Then f is convex.*

Proof. We may assume $\dim(C) = n$, by replacing \mathbb{R}^n by a plane of the appropriate dimension. It suffices to verify (3.1.1.1) for any $x, y \in C$ and any $t \in [0, 1]$. By applying a change of basis in $\text{GL}_n(\mathbb{Z})$, we may reduce to the case where the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ belong to $\angle_x C$.

We now choose $x'_1, \dots, x'_n > 0$ in turn so that for $i = 1, \dots, n$, $x_i + x'_i - y_i \in \mathbb{Q}$, $x + x'_1 \mathbf{e}_1 + \dots + x'_i \mathbf{e}_i \in \text{int}(C)$, and

$$\begin{aligned} |f(x + x'_1 \mathbf{e}_1 + \dots + x'_i \mathbf{e}_i) - f(x + x'_1 \mathbf{e}_1 + \dots + x'_{i-1} \mathbf{e}_{i-1})| &< \epsilon/n \\ |f(t(x + x'_1 \mathbf{e}_1 + \dots + x'_i \mathbf{e}_i) + (1-t)y) - f(t(x + x'_1 \mathbf{e}_1 + \dots + x'_{i-1} \mathbf{e}_{i-1}) + (1-t)y)| &< \epsilon/n. \end{aligned}$$

Namely, given x'_1, \dots, x'_{i-1} , the eligible choices of x'_i form a dense subset of an open interval with left endpoint 0. (Here we are using the continuity of the restriction of f to TRP sets of dimension 1.)

Put $x' = x + x'_1 \mathbf{e}_1 + \dots + x'_n \mathbf{e}_n$. Since $x' - y \in \mathbb{Q}^n$, the segment from x' to y is TRP. Hence

$$tf(x') + (1-t)f(y) \geq f(tx' + (1-t)y),$$

and so

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y) - 2\epsilon.$$

Since ϵ was arbitrary, this implies (3.1.1.1), yielding convexity of f . \square

Definition 3.2.7. Let C be a TRP subset of \mathbb{R}^n . For $f : C \rightarrow \mathbb{R}$ a convex function, $x \in C$, and $z \in \angle_x C$, define $f'(x, z)$ to be the directional derivative of f at x in the direction of z , i.e.,

$$f'(x, z) = \lim_{t \rightarrow 0^+} \frac{f(x + tz) - f(x)}{t}.$$

Note that this is a limit taken over a decreasing sequence; for it to exist in all cases, we must allow it to take the value $-\infty$.

Lemma 3.2.8. Let C be a TRP subset of \mathbb{R}^n , and let $f : C \rightarrow \mathbb{R}$ be a convex function. For any fixed $x \in C$, the function $z \mapsto f'(x, z)$ is convex as a function from $\angle_x C$ to $\mathbb{R} \cup \{-\infty\}$ (in the sense of satisfying (3.1.1.1)).

Proof. Take any $z_1, z_2 \in \angle_x C$. We assume first that $f'(x, z_1), f'(x, z_2) > -\infty$. Pick $u \in [0, 1]$ and put $z_3 = uz_1 + (1-u)z_2$. Given $\epsilon > 0$, choose $t > 0$ for which

$$x + tz_i \in C \quad (i = 1, 2, 3), \quad f'(x, z_i) \geq \frac{f(x + tz_i) - f(x)}{t} - \epsilon \quad (i = 1, 2).$$

Then

$$\begin{aligned} uf'(x, z_1) + (1-u)f'(x, z_2) &\geq u \frac{f(x + tz_1) - f(x)}{t} + (1-u) \frac{f(x + tz_2) - f(x)}{t} - \epsilon \\ &\geq \frac{f(uz_1 + (1-u)z_2) - f(x)}{t} - \epsilon \\ &\geq f'(x, z_3) - \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this proves the claim when both $f'(x, z_1)$ and $f'(x, z_2)$ are not $-\infty$. If one of them is $-\infty$, the same argument would imply that $f'(x, z_3) = -\infty$; this completes the proof. \square

Lemma 3.2.9. *Assume that Theorem 3.2.4 holds for compact C with n replaced by $n - 1$. Let C be a compact TRP subset of \mathbb{R}^n , and let $f : C \rightarrow \mathbb{R}$ be a function which is convex transintegral polyhedral in dimension 1. Then for any $x \in C$, the function $z \mapsto f'(x, z)$ on $\angle_x C$ is itself convex transintegral polyhedral in dimension 1.*

Proof. By Lemma 3.2.6, f is convex. By Lemma 3.2.8, $f'(x, z)$ is convex on $\angle_x C$, hence continuous on $\text{int}(\angle_x C)$. By hypothesis, for $z \in \angle_x C \cap \mathbb{Q}^n$, $f'(x, z) \in \mathbb{Z}z_1 + \cdots + \mathbb{Z}z_n$. By Theorem 3.2.2, $f'(x, z)$ is integral polyhedral on any bounded RP subset of $\text{int}(\angle_x C)$.

By subdividing C by hyperplanes, we may reduce to the case where $\angle_x C$ admits a bounded cross-section by a rational hyperplane. Pick any $z \in \angle_x C$ and $a \in \mathbb{Q}^n$ such that the set $I_{z,a} = \{u \in \mathbb{R} : z + ua \in \angle_x C\}$ is bounded. We must show that the function $g(u) = f'(x, z + ua)$ is continuous, convex, and transintegral polyhedral on $I_{z,a}$. (This suffices because we can recover all values of $f'(x, z)$ from the values on a bounded cross-section by a rational hyperplane.) By what we know about f' , we already know all of these on $\text{int}(I_{z,a})$. Consequently, it suffices to check that g is affine in a neighborhood of an endpoint of $I_{z,a}$.

For this, we may assume that the endpoint in question is a left endpoint at $u = 0$. Then z lies on the boundary of $\angle_x C$, so we can choose a codimension 1 facet D of C containing x , such that the ray from x in the direction of z has nontrivial intersection with D . By the hypothesis that Theorem 3.2.4 holds on compact TRP subsets of dimension $n - 1$, the restriction of f to D must be transintegral polyhedral. In particular, we can rescale z so that for $t \in [0, 1 + \epsilon]$ for some $\epsilon > 0$, $x + tz \in C$ and $f(x + tz) = f(x) + tf'(x, z)$.

Consider the function $h(t) = f'(x + tz, a)$ for $t \in [0, 1 + \epsilon]$. Since the difference quotient $(f(x + tz + ua) - f(x + tz))/u$ is convex in t (the term $f(x + tz + ua)$ is convex, the term $-f(x + tz)$ is affine, and dividing by u has no effect), so is $h(t)$. However, $h(t) \in \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$ for all t . This means that for $t \in (0, 1 + \epsilon)$, $h(t)$ is continuous but takes values in a discrete subset of \mathbb{R} ; this can only happen if $h(t)$ is equal to a constant value c on $(0, 1 + \epsilon)$.

Rescale a if necessary so that $x + z + a \in C$ and $f(x + z + ua) = f(x) + f'(x, z) + uc$ for $u \in [0, 1]$. We now claim that $f(x + tz + ua) = f(x) + tf'(x, z) + uc$ for $t \in [0, 1], u \in [0, t]$. Since equality holds at $(t, u) = (0, 0), (1, 0), (1, 1)$, we have by convexity of f that $f(x + tz + ua) \leq f(x) + tf'(x, z) + uc$ in the entire region. On the other hand, for any $t \in [0, 1]$, the function $f(x + tz + ua)$ in u is convex, and equals $f(x) + tf'(x, z) + uc$ for u in a right neighborhood of 0. Consequently, $f(x + tz + ua) \geq f(x) + tf'(x, z) + uc$ for $u \in [0, t]$, yielding the desired equality.

We may rewrite the last claim as $f(x + tz + tua) = f(x) + tf'(x, z) + tuc$ for $t \in [0, 1], u \in [0, 1]$. From this, we may deduce that $g(u) = f'(x, z + ua) = f'(x, z) + uc$ for $u \in [0, 1]$. This proves affinity of g near an endpoint, completing the argument. \square

We now establish the compact case of Theorem 3.2.4.

Lemma 3.2.10. *The conclusion of Theorem 3.2.4 holds if C is compact.*

Proof. We may assume that C has nonempty interior, by replacing \mathbb{R}^n by a plane containing C of the appropriate dimension. With this extra hypothesis, we proceed by induction on n , with trivial base case $n = 1$.

We have convexity of f by Lemma 3.2.6. It thus suffices to prove that f is transintegral polyhedral (and hence continuous) in a neighborhood of any $x \in C$. By Lemma 3.2.9, the restriction of $f'(x, z)$ to any compact TRP subset of $\angle_x C$ is convex transintegral polyhedral in dimension 1. By applying the induction hypothesis to the intersection of $\angle_x C$ with a rational hyperplane, we may deduce that $f'(x, z)$ is continuous, convex, and transintegral polyhedral. By Theorem 3.2.2, $f'(x, z)$ is in fact integral polyhedral.

To prove that f is transintegral polyhedral in a neighborhood of x , it suffices to do so after cutting C into finitely many pieces. We may thus reduce to the case where $f'(x, z)$ is affine on $\angle_x C$. Since $\angle_x C$ is a rational polyhedral cone, we may pick $z_1, \dots, z_l \in \angle_x C \cap \mathbb{Q}^n$ such that $\angle_x C$ is the convex hull of the rays from 0 through z_1, \dots, z_l . We may then rescale z_1, \dots, z_l so that $f(x + tz_i) = f(x) + tf'(x, z_i)$ for $i = 1, \dots, l$ and $t \in [0, 1]$.

For any z in the convex hull of z_1, \dots, z_l , we now deduce (using the affinity of $f'(x, z)$) that $f(x + z) \leq f(x) + f'(x, z)$. Since $f(x + tz)$ is convex in t , this is only possible if $f(x + tz) = f(x) + tf'(x, z)$ for $t \in [0, 1]$. We conclude that f agrees with an integral affine functional on the convex hull of $x, x + z_1, \dots, x + z_l$. As noted above, this completes the proof. \square

We now allow allow C which are no longer necessarily bounded.

Definition 3.2.11. Let C be a TRP subset of \mathbb{R}^n . Define the *small cone* of C at x , denoted $\angle'_x C$, to be the set of $z \in \mathbb{R}^n$ such that $x + tz \in C$ for all $t > 0$; this is again a convex rational polyhedral cone in \mathbb{R}^n . Moreover, it does not depend on x by the following reasoning. Write $C = \{x \in \mathbb{R}^n : \lambda_1(x), \dots, \lambda_m(x) \geq 0\}$ for some transintegral affine functionals $\lambda_1, \dots, \lambda_m$. Write $\lambda_i(x) = \lambda_{i,0}(x) + c_i$ with $\lambda_{i,0}$ linear. Then $z \in \angle'_x C$ if and only if $x \in C$ and $\lambda_{i,0}(z) \geq 0$ for $i = 1, \dots, m$. In particular, $\angle'_x C$ does not depend on the choice of $x \in C$; we thus notate it also by $\angle' C$.

Lemma 3.2.12. *The conclusion of Theorem 3.2.4 holds.*

Proof. We may again assume that C has nonempty interior in \mathbb{R}^n ; by slicing C with hyperplanes, we may further assume that the small cone $\angle' C$ is strictly convex (i.e., $\angle' C \cap -\angle' C = \{0\}$). We now induct on n , where we may assume $n \geq 2$ because the case $n = 1$ is trivial. By the induction hypothesis, the restriction of f to each boundary facet of C is convex transintegral polyhedral.

As in the proof of Lemma 3.2.9, for each boundary facet D of C , each $i \in \{1, \dots, n\}$, and each $a \in \mathbb{Q}^n$, the function $x \mapsto f'(x, a)$ is constant on the interior of each domain of affinity of the restriction of f to D . In particular, for $x \in D$ outside of a set of measure zero, $f'(x, a)$ takes only finitely many values.

By Lemma 3.2.10, f is polyhedral on any compact TRP subset of C . In particular, C is covered by domains of affinity of f ; to prove that f is polyhedral on all of C , it suffices to show that C can be covered by finitely many domains of affinity of f (see Remark 3.1.6). By Lemma 3.1.4, it suffices to check that the ambient functionals on domains of affinity of f can have only finitely many slopes.

Let U be a domain of affinity of f with ambient functional λ . Choose a basis a_1, \dots, a_n of \mathbb{Q}^n none of whose elements is contained in $\angle' C \cup (-\angle' C)$ (this is possible because $\angle' C$ is

strictly convex and $n \geq 2$). For $x \in U$ and $i \in \{1, \dots, n\}$, the function $f(x + ta_i)$ on I_{x,a_i} is convex transintegral polyhedral, so has a limiting slope at each endpoint of I_{x,a_i} . (Note that our hypothesis that $a_i \notin \angle' C \cup (-\angle' C)$ ensures that I_{x,a_i} is compact.) By the previous paragraph, for x away from a set of measure zero, these limiting slopes are themselves confined to a finite set. Since f is convex, the slope of $f(x + ta_i)$ at $t = 0$ is now also constrained to a finite set. This conclusion for $i = 1, \dots, n$ constrains the slope of λ to a finite set, proving the claim. \square

3.3 Variation of subsidiary radii

In this subsection, we will extend Theorem 2.4.4 into a higher-dimensional generalization (Theorem 3.3.9). We keep Hypothesis 2.0.1 and Notation 2.0.2. We begin by introducing the setup of [10, Section 4.1].

Notation 3.3.1. Throughout this subsection, we put $I = \{1, \dots, n\}$ for notational simplicity.

Notation 3.3.2. For X an n -tuple:

- for A an $n \times n$ matrix, write X^A for the n -tuple whose j -th entry is $\prod_{i=1}^n x_i^{A_{ij}}$;
- for c a number, put $X^c = (x_1^c, \dots, x_n^c)$.

Definition 3.3.3. For a subset $C \subset \mathbb{R}^n$, let e^{-C} denote the subset $\{e^{-r_I} : r_I \in C\} \subset (0, +\infty)^n$. A subset S of $[0, +\infty)^n$ is *log-(T)RP* if S is the closure of $\overset{\circ}{S} = e^{-C}$ for some (T)RP subset C of \mathbb{R}^n . We say S is *ind-log-(T)RP* if it is a union of an increasing sequence of log-(T)RP sets S_α ; we denote $\overset{\circ}{S} = \cup_\alpha \overset{\circ}{S}_\alpha$. For instance, any open subset of $[0, +\infty)^n$ is covered by ind-log-RP subsets.

Caution 3.3.4. The subset $(0, 1]$ is an ind-log-TRP subset but not a log-TRP subset. By contrast, $[0, 1]$ is a log-TRP subset.

Definition 3.3.5. Let $C \subset \mathbb{R}^n$ be a TRP subset defined by (3.1.5.1), where $\lambda_s(x_I) = a_{s,1}x_1 + \dots + a_{s,n}x_n + b_s$ for $a_{s,i} \in \mathbb{Z}$ and $s = 1, \dots, r$. Denote the closure of e^{-C} in $[0, +\infty)^n$ by S . Define $A_K(S)$ to be the subspace of the (Berkovich) analytic n -space with coordinates t_1, \dots, t_n satisfying the condition $(|t_1|, \dots, |t_n|) \in S$. Precisely,

$$\Gamma(A_K(S), \mathcal{O}) = K \langle t_I^{a_{1,I}} / e^{-b_1}, \dots, t_I^{a_{r,I}} / e^{-b_r} \rangle.$$

For an ind-log-TRP subset $S = \cup_\alpha S_\alpha$, we define $A_K(S) = \cap_\alpha A_K(S_\alpha)$.

Definition 3.3.6. Let S be an ind-log-TRP subset of $[0, +\infty)^n$. A $(\partial_{I \cup J})$ -differential module M over $X = A_K(S)$ is a locally free coherent sheaf together with an integrable connection

$$\nabla : M \rightarrow M \otimes \left(\bigoplus_{j=1}^m \mathcal{O}_X \cdot du_j \oplus \bigoplus_{i=1}^n \mathcal{O}_X \cdot dt_i \right).$$

We label the derivations $\partial_1, \dots, \partial_m$ as usual, and put $\partial_{m+1} = \partial_{t_1}, \dots, \partial_{m+n} = \partial_{t_n}$.

Notation 3.3.7. For $\eta_I = (\eta_1, \dots, \eta_n) \in \overset{\circ}{S}$, let F_{η_I} be the completion of $K(t_I)$ with respect to the η_I -Gauss norm. Write $f_l(M, r_I) = -\log IR(M \otimes F_{e^{-r_I}}; l)$ and $F_l(M, r_I) = f_1(M, r_I) + \dots + f_l(M, r_I)$ for $l = 1, \dots, \text{rank } M$.

Lemma 3.3.8. Given $\eta_I \in (0, +\infty)^n$ and $A \in \text{GL}_n(\mathbb{Z})$, let M be a differential module over $F_{\eta_I^A}$, and let $h_A^* : F_{\eta_I^A} \rightarrow F_{\eta_I}$ be given by $t_I \mapsto t_I^A$. Then $IR(M) = IR(h_A^* M)$.

Proof. This follows from [10, Proposition 4.2.7] (which is itself an immediate consequence of [10, Lemma 4.1.5]) applied to A and A^{-1} . \square

Theorem 3.3.9. Let S be an ind-log-TRP subset of $[0, +\infty)^n$, and let M a differential module of rank d over $A_K(S)$.

- (a) (Continuity) For $l = 1, \dots, d$, the functions $f_l(M, r_I)$ and $F_l(M, r_I)$ are continuous.
- (b) (Convexity) For $l = 1, \dots, d$, the function $F_l(M, r_I)$ is convex.
- (c) (Polyhedrality) For $r_I \in -\log \overset{\circ}{S}$, if $l = d$ or $f_l(M, r_I) > f_{l+1}(M, r_I)$, then $F_l(M, r_I)$ is transintegral polyhedral in some neighborhood of r_I . Moreover, on any TRP subset of $-\log \overset{\circ}{S}$, $d!F_l(M, r_I)$ and $F_d(M, r_I)$ are transintegral polyhedral functions.
- (d) (Monotonicity) Assume that S is log-TRP. Then for any $r_I, r'_I \in -\log \overset{\circ}{S}$, if $r_i \leq r'_i$ for $i \in I$ and $(1-t)r_I + tr'_I \in -\log \overset{\circ}{S}$ for any $t \in [0, +\infty)$, then $F_l(M, r_I) \geq F_l(M, r'_I)$ for $l = 1, \dots, d$.

Proof. We first prove (a)-(c). We need only verify that, for $l = 1, \dots, d$, $d!F_l(M, r_I)$ and $F_d(M, r_I)$ satisfy the conditions of Theorem 3.2.4. Moreover, by translating and enlarging K if necessary, it suffices to check the hypothesis of Theorem 3.2.4 for $I_{x,a}$ in the case $x = 0$.

It suffices to consider $a = a_I \in \mathbb{Z}^n$ with $\gcd(a_I) = 1$. Let us describe $f_l(M, a_I t)$ and $F_l(M, a_I t)$ for $l = 1, \dots, d$ and $t \in I_{0,a_I}$. Pick an $n \times n$ invertible integral matrix A with (a_I) as the first row. Equip $A_K(S^{A^{-1}})$ with the coordinates (s_I) , and define the toroidal transform $\phi : A_K(S^{A^{-1}}) \rightarrow A_K(S)$ by $\phi^*(t_I) = s_I^A$, where $S^{A^{-1}} = \{X^{A^{-1}} | X \in S\}$. By Lemma 3.3.8, $f_l(M, a_I t) = f_l(\phi^* M, (a_I A^{-1})t)$. The theorem follows from Theorem 2.4.4.

To prove (d), by continuity, we may assume that $r_I - r'_I$ are all rational numbers. By an argument as in the previous paragraph, we may reduce to the one-dimensional case. In this case, we get a differential module over a disc, so the desired statement follows from Theorem 2.4.4(c). \square

3.4 Decomposition by subsidiary radii

To conclude, we extend the theorems of §2.5 to higher-dimensional spaces.

Lemma 3.4.1. Suppose $r \in \{0, \dots, n\}$. Put $C = \{(x_I) | x_I \geq 0, x_1 + \dots + x_r \leq 1\} \subset \mathbb{R}^n$, and let C_ϵ be any TRP subset of \mathbb{R}^n containing C in its interior. Let S (resp. S_ϵ) denote the closure of e^{-C} (resp. e^{-C_ϵ}) in $[0, +\infty)^n$, which is a log-TRP subset. Let M be a differential

module of rank d over $A_K(S_\epsilon)$. Suppose that the following conditions hold for some $l \in \{1, \dots, d-1\}$.

(a) The function $F_l(M, r_I)$ is affine for $(r_I) \in C_\epsilon$.

(b) We have $f_l(M, r_I) > f_{l+1}(M, r_I)$ for $(r_I) \in C_\epsilon$.

Then M admits a unique direct sum decomposition over $A_K(S)$ separating the first l subsidiary radii of $M \otimes F_{e^{-r_I}}$ for any $(r_I) \in C$.

Proof. Note that $\Gamma(A_K(S), \mathcal{O}) = K\langle t_I, e^{-1}/t_1 \cdots t_r \rangle$ may be embedded into the completion $F_{1, \dots, 1}$ of $K(t_1, \dots, t_n)$ for the $(1, \dots, 1)$ -Gauss norm. For $i = 1, \dots, n$, let $F_{1, \dots, 1}^{(i)}$ be the completion of $K(t_1, \dots, \widehat{t_i}, \dots, t_n)$ for the $(1, \dots, 1)$ -Gauss norm; then the image of $\Gamma(A_K(S), \mathcal{O})$ also belongs to each of the subrings

$$F_{1, \dots, 1}^{(i)} \langle e^{-1}/t_i, t_i \rangle \quad (i = 1, \dots, r); \quad F_{1, \dots, 1}^{(i)} \langle t_i \rangle \quad (i = r+1, \dots, n),$$

In fact, it is equal to the intersection of these subrings; this is true because C is the convex hull of the union of the segments

$$\begin{aligned} \{(x_1, \dots, x_n) : 0 \leq x_i \leq 1; \quad x_j = 0 \quad (j \neq i)\} & \quad (i = 1, \dots, r) \\ \{(x_1, \dots, x_n) : 0 \leq x_i; \quad x_j = 0 \quad (j \neq i)\} & \quad (i = r+1, \dots, n). \end{aligned}$$

Consequently, by Lemma 2.3.2, it suffices to prove the decomposition over the rings $F_{1, \dots, 1}^{(i)} \langle e^{-1}/t_i, t_i \rangle$ for $i = 1, \dots, r$ and $F_{1, \dots, 1}^{(i)} \langle t_i \rangle$ for $i = r+1, \dots, n$. The former case follows by applying Theorem 2.5.1 to $M \otimes F_{1, \dots, 1} \langle e^{-1-\epsilon}/t_i, t_i/e^\epsilon \rangle$ for $i = 1, \dots, r$ for some $\epsilon > 0$; the latter case follows by applying Theorem 2.5.2 to $F_{1, \dots, 1} \langle t_i/e^\epsilon \rangle$ for $i = r+1, \dots, n$ for some $\epsilon > 0$. \square

Theorem 3.4.2. *Let S be a ind-log-TRP subset of $[0, +\infty)^n$, and let M a differential module of rank d over $A_K(\text{int}(S))$. Suppose that the following conditions hold for some $l \in \{1, \dots, d-1\}$.*

(a) The function $F_l(M, r_I)$ is affine for $(r_I) \in \text{int}(-\log \overset{\circ}{S})$.

(b) We have $f_l(M, r_I) > f_{l+1}(M, r_I)$ for $(r_I) \in \text{int}(-\log \overset{\circ}{S})$.

Then M admits a unique direct sum decomposition over $A_K(\text{int}(S))$ separating the first l subsidiary radii of $M \otimes F_{e^{-r_I}}$ for any $(r_I) \in \text{int}(-\log \overset{\circ}{S})$.

Proof. We can cover $\text{int}(S)$ by log-TRP subsets $S_\alpha \subset \text{int}(S)$ such that for each point of $x \in \text{int}(S)$, there exists a neighborhood of x contained in some S_α . Moreover, we can choose those S_α to be simplicial, i.e., under a toroidal transform and rescaling, each S_α can be transformed into the form desired for Lemma 3.4.1. Since S_α lies in the interior of S , the decomposition follows from Lemma 3.4.1 by gluing the decompositions obtained on each of the S_α . \square

Lemma 3.4.3. *Suppose $r \in \{0, \dots, n\}$. Put $C = \{(x_I) | x_I \geq 0, x_1 + \dots + x_r < 1\} \subset \mathbb{R}^n$, and let C_ϵ be any TRP subset of \mathbb{R}^n containing C in its interior. Let S_ϵ denote the closure of e^{-C_ϵ} in $[0, +\infty)^n$, which is a log-TRP subset. Let S be the set of points $(s_I) \in S_\epsilon$ such that $s_I \leq 1$ and $s_1 \cdots s_r > e^{-1}$. Let R be the subring of $\Gamma(A_K(S_\epsilon), \mathcal{O})$ consisting of those f for which $|f|_{s_I}$ is bounded over $(s_I) \in S$. Let M be a differential module of rank d over $A_K(S_\epsilon)$. Suppose that the following conditions hold for some $l \in \{1, \dots, d-1\}$.*

(a) *The function $F_l(M, r_I)$ is affine for $(r_I) \in C_\epsilon$.*

(b) *We have $f_l(M, r_I) > f_{l+1}(M, r_I)$ for $(r_I) \in C_\epsilon$.*

Then $M \otimes R$ admits a unique direct sum decomposition separating the first l subsidiary radii of $M \otimes F_{e^{-r_I}}$ for any $(r_I) \in C$.

Proof. Let F be the completion of $\text{Frac } R$ for the $(1, \dots, 1)$ -Gauss norm. Define $F_{1, \dots, 1}^{(i)}$ as in the proof of Lemma 3.4.1. Then inside F , R is the intersection of the rings

$$F_{1, \dots, 1}^{(i)} \langle 1/t_i^{-1}, t_i^{-1}/e \rangle_0 \quad (i = 1, \dots, r); \quad F_{1, \dots, 1}^{(i)} \langle t_i \rangle \quad (i = r+1, \dots, n).$$

We may thus argue as in Lemma 3.4.1, but using Theorem 2.5.5 instead of Theorems 2.5.1 and 2.5.2. \square

Theorem 3.4.4. *Let S be a log-TRP subset of $[0, +\infty)^n$. Let R be the subring of $\Gamma(A_K(\text{int}(S)), \mathcal{O})$ consisting of those f for which $|f|_{s_I}$ is bounded over $s_I \in \text{int}(S)$. Let M be a differential module of rank d over $A_K(S)$. Suppose that the following conditions hold for some $l \in \{1, \dots, d-1\}$.*

(a) *The function $F_l(M, r_I)$ is affine for $(r_I) \in -\log \overset{\circ}{S}$.*

(b) *We have $f_l(M, r_I) > f_{l+1}(M, r_I)$ for $(r_I) \in -\log \overset{\circ}{S}$.*

Then $M \otimes R$ admits a unique direct sum decomposition separating the first l subsidiary radii of $M \otimes F_{e^{-r_I}}$ for any $(r_I) \in \text{int}(-\log \overset{\circ}{S})$.

Proof. Analogous to Theorem 3.4.2, except using Lemma 3.4.3 instead of Lemma 3.4.1. \square

Remark 3.4.5. It may be helpful to illustrate the argument needed to reduce Theorem 3.4.4 to Lemma 3.4.3 with an explicit example. Take $S = [0, 1]^2$, so that $R = \mathfrak{o}_K[[x, y]] \otimes_{\mathfrak{o}_K} K$. We must partition $\text{int}(-\log \overset{\circ}{S}) = (0, +\infty)^2$ into regions to which Lemma 3.4.3 may be applied. One such partition consists of

$$\begin{aligned} &\{(x, y) \in \mathbb{R}^2 : 0 < x, 0 < y \leq \min\{x, 1\}\}, \\ &\{(x, y) \in \mathbb{R}^2 : 0 < y, 0 < x \leq \min\{y, 1\}\}, \\ &\{(x, y) \in \mathbb{R}^2 : 1 \leq x, 1 \leq y\}. \end{aligned}$$

Since the parts all contain $(1, 1)$, we can glue the three resulting decompositions together by matching them on $M \otimes F_{e^{-1}, e^{-1}}$.

Remark 3.4.6. Note that Lemma 3.4.3 is not a special case of Theorem 3.4.4. We leave the formulation and proof of a common generalization as a somewhat awkward exercise for the reader.

Remark 3.4.7. By Remark 2.5.7, in Theorem 3.4.4, if $\log |K^\times| \subseteq \mathbb{Q}$ and $-\log \mathring{S}$ is RP, we may also take M to be defined over R . For example, if K carries the trivial valuation (forcing $p = 0$) and

$$S = \{(x, y) \in (0, 1]^2 : xy = e^{-1}\},$$

then $R = K[[x, y]][x^{-1}, y^{-1}]$. This example can be used in the study of good formal structures for flat holomorphic connections; however, one needs to refine Theorem 3.4.4 slightly in case $p = 0$, to remove the need for strict inequality on the boundary of $-\log \mathring{S}$. For this, we defer to [13].

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